Advanced Mechanics of Mechanical Systems

Lectures:
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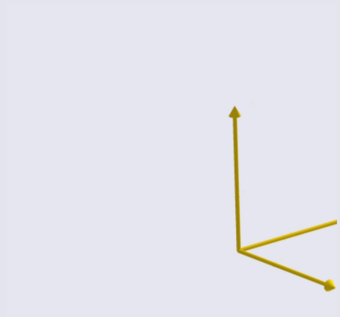


Welcome to Aalborg University!

- **Venues**

Fib. 14, room 59, Fib. 16, room 1.111, Canteen, No 2 Bus stop, Pizza place

http://www.info.aaumap.portal.aau.dk/adresser

M. S. Andersen: Advanced Mechanics of Mechanical Systems

Practical information

- **Food:**
  - Rolls and coffee will be served every morning.
  - Cake and coffee will be served every afternoon.
  - Lunch:
    - Sandwishes, hot dishes and salad can be purchased in the AAU canteen.
    - Pizza, Pasta and more at Bella Italia.
  - Social event:
    - Dinner at a restaurant down town thursday the 20th. More information will follow later.

- **Wireless internet**:
  - Eduroam
    - Use your own credentials.
  - AAU-1-DAY.
    - New password every day. We will provide these for you.

Course outline

Day 1: (Tue., 18/09): Fundamentals of kinematics of rigid multi-body systems

M. S. Andersen (Fib. 14, Room 59 and Fib.16, Room 1.111)

Day 2: (Wed., 19/09): Dynamic modeling of rigid multi-body systems

S. Bai (Fib 14, Room 59)

Day 3: (Fri., 21/09) Introduction to flexible system dynamics

A. Mikkola (Fib 14, Room 59)

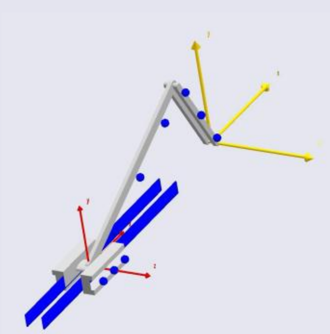
Workshop: (Fri. 20/06) Dynamic modeling with Adams

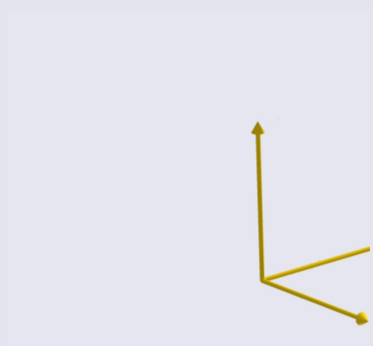
IdéPro (Fib 14, room 59)

NB! Bring your labtop for this part of the course

Multi-body mechanical system

- A multi-body mechanical system consists of **bodies**, **joints**, **actuators** and **loads.**
- The aim of the model is to either compute motion or forces.





Degrees of freedom

Degrees of freedom (DOF): are the independent ways of motion for a mechanical system. T*he number of DOF* gives the minimum number of coordinates required to fully describe the motion.

Counting the number of DOF

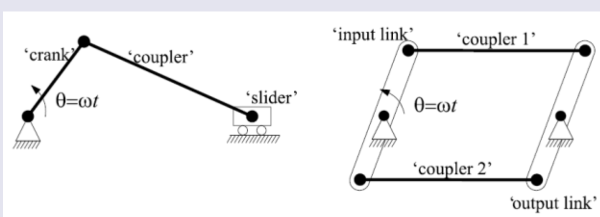
$$n^{\text{DOF}} = 6n^{\text{bodies}} - \sum_{\text{joints}} n^{\text{constraints, 3D}} \quad \text{(spatial system)}$$

$$n^{\text{DOF}} = 3n^{\text{bodies}} - \sum_{\text{joints}} n^{\text{constraints, 2D}} \quad \text{(planar system)}$$

Constraints that include motion information are called *kinematic drivers* and the number of these, we denote n^{Drivers}.

Degrees of freedom

Examples



Slider-crank (2D):

3 bodies x 3 DOF	9
3 revolute joints x 2 DOF	-6
1 translational joint x 2 DOF	-2
	1 DOF

4-bar linkage (2D):

4 bodies x 3 DOF	12
6 revolute joints x 2 DOF	-12
1 translational joint x 2 DOF	0
	0 DOF

(One redundant constraint!)

M. S. Andersen: Advanced Mechanics of Mechanical Systems

Kinematic determinacy

$n^{\text{DOF}} - n^{\text{Drivers}} > 0$: Kinematically indeterminate system

This type of system is the most general in dynamics of mechanical linkages, where we have to solve the second-order differential equations of motion.

$n^{\text{DOF}} - n^{\text{Drivers}} = 0$: Kinematically determinate system

This type of system is much easier to handle. The motion is completely prescribed by the drivers, implying that kinematics and kinetics are decoupled problems.

A special case is when $n^{\text{DOF}} = n^{\text{Drivers}} = 0$, where the system is statically determinate.

$n^{\text{DOF}} - n^{\text{Drivers}} < 0$: Kinematically over-determinate system

This type of system requires that deformations must be taken into account.

Kinematic analysis

Kinematic analysis:

- Analysis of the motion of a mechanical system without consideration of the forces that cause the motion.
- Includes the description of the involved bodies, whether rigid or flexible, their connection (joints) and motion.
- Aimed at computing the position, velocity and acceleration relationship between the involved bodies.
- Analysis of a mechanical system purely based on kinematic information requires a kinematically determinate system.

Math preliminaries

Geometric vectors, denoted with an arrow:

$$\vec{a} = X\vec{I} + Y\vec{J} + Z\vec{K} = x\vec{i} + y\vec{j} + z\vec{k}$$

are vectors spanning between two points in space (either 2D or 3D).

Algebraic vectors, denoted with an underline:

$$\underline{a} = \begin{bmatrix} X & Y & Z \end{bmatrix}$$

are arrays of numbers, which do not necessarily have a geometric interpretation.

Skew-symmetric matrix of $\underline{a}$: $\underline{\underline{\tilde{a}}} = \begin{bmatrix} 0 & -Z & Y \\ Z & 0 & -X \\ -Y & X & 0 \end{bmatrix}$

Matrices are denoted with a double underline

Cross-product: $\underline{a} \times \underline{b} = \underline{\underline{\tilde{a}}}\,\underline{b}$

Matrix-vector product

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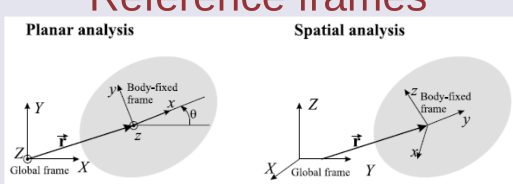
Reference frames



- *Global frame*: The global frame is our main reference, i.e. the frame where we measure the absolute motion.
- *Local frame* (body-fixed frame): A local frame is an assistive tool that allows us to consider parts of the motion separately if convenient.
- A vector or point expressed in the global frame is called a global vector or point.
- A vector or point expressed in the local frame is called a local vector or point.

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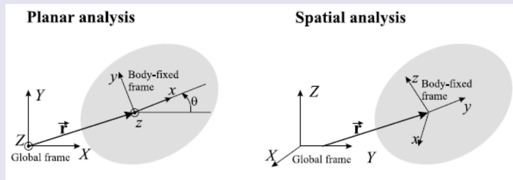
Transformation matrix



Transformation matrix

$$\underline{a} = X\underline{I} + Y\underline{J} + Z\underline{K} = \begin{bmatrix} \underline{I} & \underline{J} & \underline{K} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = x\underline{i} + y\underline{j} + z\underline{k} = \begin{bmatrix} \underline{i} & \underline{j} & \underline{k} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underline{\underline{A}}\underline{a}'$$

Global vector

Local vector

Properties (orthonormal matrix):

$$\underline{\underline{A}}^T \underline{\underline{A}} = I \text{ and } \underline{\underline{A}}^{-1} = \underline{\underline{A}}^T$$

Inverse relationship:

$$\underline{a}' = \underline{\underline{A}}^T \underline{a}$$

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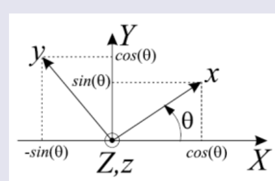
The transformation matrix in 2D

Definition of the rotational coordinate in 2D is straight forward.



The transformation matrix:

$$\underline{\underline{A}} = \begin{pmatrix} \underline{i} & \underline{j} \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

The time derivative of $\underline{\underline{A}}$:

$$\dot{\underline{\underline{A}}} = \underline{\underline{B}}\dot{\theta} \quad \text{where} \quad \underline{\underline{B}} = \frac{\partial \underline{\underline{A}}}{\partial \theta} = \begin{pmatrix} -\sin\theta & -\cos\theta \\ \cos\theta & -\sin\theta \end{pmatrix}$$

The time derivative of $\underline{\underline{B}}$:

$$\dot{\underline{\underline{B}}} = -\underline{\underline{A}}\dot{\theta}$$

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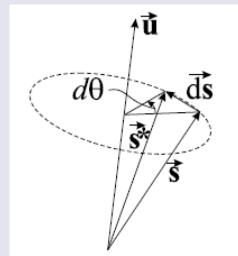
Angular velocity vector

Rotational velocity

$$\vec{\omega} = \dot{\theta}\vec{u}$$

Instantaneous rotation axis



It can be proven that the angular velocity vector is a *geometric vector*. I.e. for instance the relative angular velocity can be computed using vector differences.

Velocity of the vector $\vec{s}$:

$$\dot{\vec{s}} = \vec{\omega}\times\vec{s} = \underline{\underline{\tilde{\omega}}}\,\underline{s}$$

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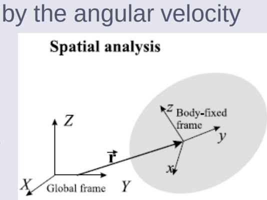
Relationship between the transformation matrix and the angular velocity vector

Assume that a body is rotating as described by the angular velocity vector, $\vec{\omega}$. Hence, each axis is rotating.



The velocity of the axis are:

$$\dot{\vec{i}} = \vec{\omega}\times\vec{i} = \underline{\underline{\tilde{\omega}}}\,\underline{i} \qquad \dot{\vec{j}} = \vec{\omega}\times\vec{j} = \underline{\underline{\tilde{\omega}}}\,\underline{j} \qquad \dot{\vec{k}} = \vec{\omega}\times\vec{k} = \underline{\underline{\tilde{\omega}}}\,\underline{k}$$

Hereby, we can express the time-derivative of the transformation matrix:

$$\dot{\underline{\underline{A}}} = \left[\dot{\underline{i}} \quad \dot{\underline{j}} \quad \dot{\underline{k}}\right] = \left[\underline{\underline{\tilde{\omega}}}\,\underline{i} \quad \underline{\underline{\tilde{\omega}}}\,\underline{j} \quad \underline{\underline{\tilde{\omega}}}\,\underline{k}\right] = \underline{\underline{\tilde{\omega}}}\left[\underline{i} \quad \underline{j} \quad \underline{k}\right] = \underline{\underline{\tilde{\omega}}}\,\underline{\underline{A}} = \underline{\underline{A}}\,\underline{\underline{\tilde{\omega}}}'$$

where $\underline{\omega}'$ is angular velocity vector in local coordinates.

This we are going to use when deriving constraint equations.

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Rotational coordinates in 3D

- So far, we have described the orientation of the bodies by means of a rotation matrix and the angular velocity.
- The rotation matrix: 9 parameters and 6 orthonormality constraints.
- Other options:
  - Cardan angles.
  - Cartesian rotation vector.
  - Euler parameters.
  - And more.

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Rotational coordinates in 3D

- Ideally, a good parametrization of rotation must:
  - Facilitate geometric interpretation.
  - Be convenient for algebraic manipulations.

  } Theoretical advantages

  - Be as linear as possible in terms of the rotation angle(s).
  - Present a minimal (3) number of parameters.

  } Computational advantages

  - Not lead to singularities:
    - in the definition.
    - in the inverse problem.
    - in the tangent matrix.
    - Notice that *all* rotational coordinates in 3D with only three parameters will always include a singularity.

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Cardan angles

- Rotation matrices for rotation around each axis:

$$\underline{\underline{A}}_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_x & -\sin\theta_x \\ 0 & \sin\theta_x & \cos\theta_x \end{bmatrix}, \quad \underline{\underline{A}}_y = \begin{bmatrix} \cos\theta_y & 0 & \sin\theta_y \\ 0 & 1 & 0 \\ -\sin\theta_y & 0 & \cos\theta_y \end{bmatrix},$$

$$\underline{\underline{A}}_z = \begin{bmatrix} \cos\theta_z & -\sin\theta_z & 0 \\ \sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

z-y-x sequence

x-y-z sequence

- Combined rotation matrix:
  $\underline{\underline{A}} = \underline{\underline{A}}_l \underline{\underline{A}}_m \underline{\underline{A}}_n$
  where *l*, *m*, *n* indicate the rotation axis and can be *x*, *y*, *z*. The rotation sequence is denoted l-m-n.
- The special cases:
  - z-x-z order is called Euler angles.
  - x-y-z order is called Bryant angles.

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Cardan angles from transformation matrix

- Rotation sequence z-y-x: $\underline{\underline{A}} = \underline{\underline{A}}_z \underline{\underline{A}}_y \underline{\underline{A}}_x$

$$\underline{\underline{A}} = \begin{bmatrix} c\theta_y c\theta_z & -c\theta_x s\theta_z + s\theta_x s\theta_y c\theta_z & s\theta_x s\theta_z + c\theta_x s\theta_y c\theta_z \\ c\theta_y s\theta_z & c\theta_x c\theta_z + s\theta_x s\theta_y s\theta_z & -s\theta_x c\theta_z + c\theta_x s\theta_y s\theta_z \\ -s\theta_y & s\theta_x c\theta_y & c\theta_x c\theta_y \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

- First compute the rotation angle around the y-axis (two possible solutions):

$$\theta_y = -\arcsin(A_{31}) \quad \vee \quad \theta_y = \pi - \arcsin(A_{31})$$

- Then compute the other two angles for each of the two solutions of :

$$\theta_z = \arctan 2(A_{21} / \cos(\theta_y), A_{11} / \cos(\theta_y))$$

$$\theta_x = \arctan 2(A_{32} / \cos(\theta_y), A_{33} / \cos(\theta_y))$$

- Determine which of the two solutions to use. For instance using the solution that is closest to the solution from a previous time step (or which is closest to zero).

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Cardan angles from transformation matrix

- Adjust the angles by 2π until they are within a desired range – for instance to ensure that the computed angles are continuous over time.
- **NB!** There are singularities for this particular rotation sequence, when
- $\cos(\theta_y) = 0 \;\; \Rightarrow \;\; \theta_y = \pm n\pi + \pi/2, \;\; n = 1,2,\ldots\infty$

Time-derivatives of Cardan angles and the angular velocity vector

- Each component of the Cardan rotation sequence contributes towards the total angular velocity.
- Relation to the global angular velocity vector:

$$\underline{\omega} = \underline{u}_l\dot{\theta}_l + \underline{\underline{A}}_l\underline{u}_m\dot{\theta}_m + \underline{\underline{A}}_l\underline{\underline{A}}_m\underline{u}_n\dot{\theta}_n = \underbrace{\begin{bmatrix}\underline{u}_l & \underline{\underline{A}}_l\underline{u}_m & \underline{\underline{A}}_l\underline{\underline{A}}_m\underline{u}_n\end{bmatrix}}_{\underline{\underline{T}}}\begin{bmatrix}\dot{\theta}_l\\ \dot{\theta}_m\\ \dot{\theta}_n\end{bmatrix}$$

Tangent matrix

- Computed by transforming each velocity component into the global reference frame.

- Relation to the local angular velocity vector. Obtained by pre-multiplication by $\underline{\underline{A}}^T = \underline{\underline{A}}_n^T\underline{\underline{A}}_m^T\underline{\underline{A}}_l^T$:

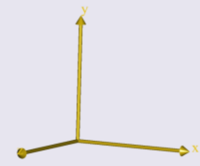
$$\underline{\omega}' = \underline{\underline{A}}^T\underline{u}_l\dot{\theta}_l + \underline{\underline{A}}_n^T\underline{\underline{A}}_m^T\underline{u}_m\dot{\theta}_m + \underline{u}_n\dot{\theta}_n = \underbrace{\begin{bmatrix}\underline{\underline{A}}^T\underline{u}_l & \underline{\underline{A}}_n^T\underline{\underline{A}}_m^T\underline{u}_m & \underline{u}_n\end{bmatrix}}_{\underline{\underline{T}}'}\begin{bmatrix}\dot{\theta}_l\\ \dot{\theta}_m\\ \dot{\theta}_n\end{bmatrix}$$

to bring the computation into the local reference frame.

Cardan angles

- Advantages:
  - Minimal set of coordinates.
  - Nice geometric interpretation.

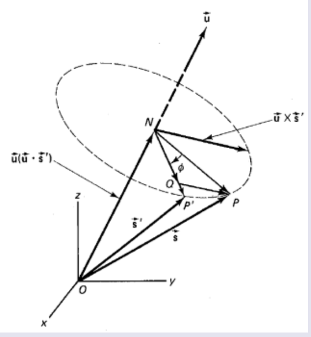
- Disadvantages:
  - Depends on rotation sequence.
  - The rotation matrix involves numerous trigonometric functions.
  - Singularities for the inverse problem.



Euler's theorem

- Euler's theorem: *The general displacement of a body with one point fixed is a rotation about some axis.*

- Rotational formula:
  $\underline{s} = \underline{s}' \cos\phi + \underline{u}(\underline{u}^T \underline{s}')(1-\cos\phi) + \underline{\underline{\tilde{u}}}\,\underline{s}' \sin\phi$

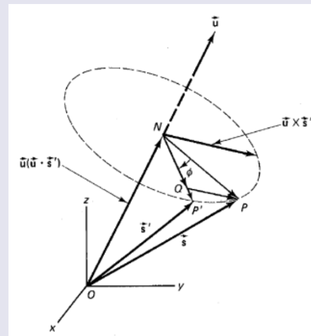


Cartesian rotation vector

$$\underline{\psi} = \phi \underline{\mathrm{u}}$$

3 parameters

Rotation angle | Rotation axis



- Rotation matrix:

$$\underline{\underline{A}} = \underline{\underline{I}} + \frac{\sin\|\underline{\psi}\|_2}{\|\underline{\psi}\|_2}\underline{\underline{\tilde{\psi}}} + \frac{1-\cos\|\underline{\psi}\|_2}{\|\underline{\psi}\|_2^2}\underline{\underline{\tilde{\psi}}}\,\underline{\underline{\tilde{\psi}}}$$

- Tangent matrix:

$$\underline{\underline{T}} = \underline{\underline{I}} + \frac{\cos\|\underline{\psi}\|_2 - 1}{\|\underline{\psi}\|_2^2}\underline{\underline{\tilde{\psi}}} + \left(1 - \frac{\sin\|\underline{\psi}\|_2}{\|\underline{\psi}\|_2}\right)\frac{\underline{\underline{\tilde{\psi}}}\,\underline{\underline{\tilde{\psi}}}}{\|\underline{\psi}\|_2^2}$$

- Angular velocity vector:

$$\underline{\omega} = \underline{\underline{T}}\,\underline{\dot{\psi}}$$

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Cartesian rotation vector

$$\underline{\underline{A}} = \underline{\underline{I}} + \frac{\sin\|\underline{\psi}\|_2}{\|\underline{\psi}\|_2}\underline{\underline{\tilde{\psi}}} + \frac{1-\cos\|\underline{\psi}\|_2}{\|\underline{\psi}\|_2^2}\underline{\underline{\tilde{\psi}}}\,\underline{\underline{\tilde{\psi}}}$$

No singularity in the definition:

$$\lim_{\|\underline{\psi}\|_2 \to 0} \underline{\underline{A}} = \lim_{\|\underline{\psi}\|_2 \to 0} \underline{\underline{T}} = \underline{\underline{I}}$$

Singularity in the inverse problem:

$$\underline{\underline{A}} = \underline{\underline{I}} \quad \text{if} \quad \|\underline{\psi}\| = 2\pi k$$

$\underline{u}$ cannot be determined

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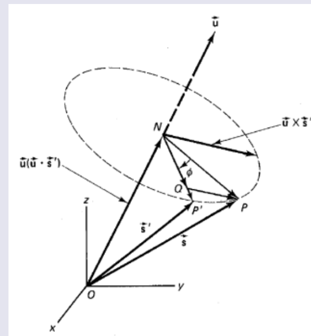
Cartesian rotation vector

- Advantages:
  - Minimal set of coordinates.
  - Easy geometric interpretation.
  - Absence of singularities in the definition.
  - Simple linearized expressions.
- Disadvantages:
  - Singularities for the inverse problem.



Euler parameters

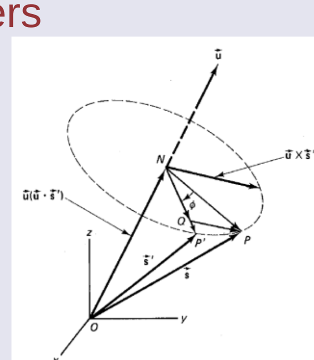
- Definition:

$$e_0 = \cos\frac{\phi}{2}$$

$$\underline{e} = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \underline{u}\sin\frac{\phi}{2}$$

- The rotational formula:

$$\underline{s} = (2e_0^2 - 1)\underline{s}' + 2\underline{e}(\underline{e}^T \underline{s}') + 2e_0(\underline{\underline{\tilde{e}}}\,\underline{s}')$$



Euler parameters

- Setting $\underline{s}'$ outside a bracket to obtain the transformation matrix:

$$\underline{s} = \underbrace{\left((2e_0^2 - 1)\underline{\underline{I}} + 2\underline{e}\,\underline{e}^T + 2e_0\underline{\underline{\tilde{e}}}\right)}_{\underline{\underline{A}}}\underline{s}'$$

- Notice that the Euler parameters are not independent and must statisfy:

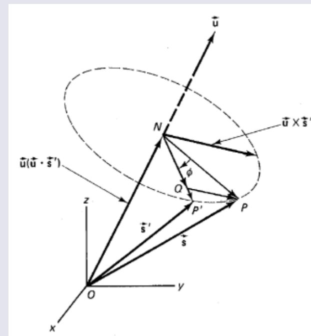
$$e_0^2 + \underline{e}^T\underline{e} = 1$$

- Also sometimes written:

$$\underline{p}^T\underline{p} = 1$$

- where

$$\underline{p} = \begin{bmatrix} e_0 & e_1 & e_2 & e_3 \end{bmatrix}^T$$



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Identities with Euler parameters

$$\underline{\underline{G}} = \begin{bmatrix} -e_1 & e_0 & -e_3 & e_2 \\ -e_2 & e_3 & e_0 & -e_1 \\ -e_3 & -e_2 & e_1 & e_0 \end{bmatrix} \qquad \underline{\underline{L}} = \begin{bmatrix} -e_1 & e_0 & e_3 & -e_2 \\ -e_2 & -e_3 & e_0 & e_1 \\ -e_3 & e_2 & -e_1 & e_0 \end{bmatrix}$$

- Transformation matrix:

$$\underline{\underline{A}} = \underline{\underline{G}}\,\underline{\underline{L}}^T$$

- Tangent matrix:

$$\underline{\underline{T}} = 2\underline{\underline{G}}, \quad \underline{\underline{T}}' = 2\underline{\underline{L}}$$

- Angular velocity:

$$\underline{\omega} = \underline{\underline{T}}\,\underline{\dot{p}}$$

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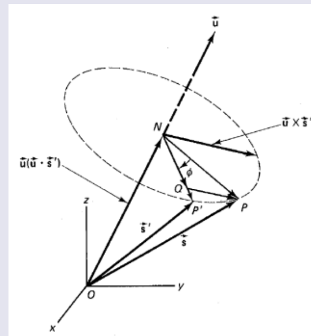
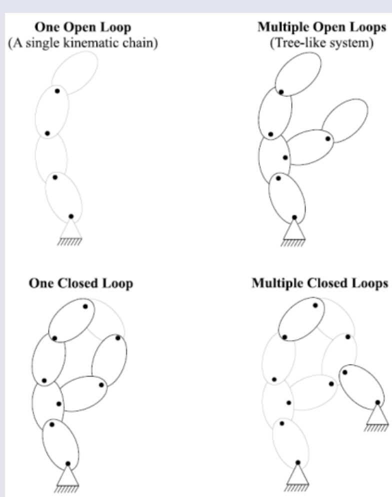
Euler parameters

- Advantages:
  - No singularities!
  - Purely algebraic quantities, i.e. no trigonomic functions.
- Disadvantages:
  - An extra constraint equations.
  - Difficult to drive.
  - More coordinates than strictly needed.



Open and closed loop systems



Open loop systems

Open-loop systems with relative coordinates (minimal set):

Open-loop systems can be handled fairly easy by defining coordinates associated with the DOF of each joint. These coordinates corresponds directly to the DOF of the system and any kinematic quantity can be computed directly from these.

Open loop systems

Position analysis:

$$\underline{\underline{A}}_j = \underline{\underline{A}}_i \underline{\underline{A}}_{j/i}$$

$$\underline{r}_j = \underline{r}_i + \underline{s}_{j/i} = \underline{r}_i + \underline{\underline{A}}_i \underline{s}^{(i)}_{j/i}$$

Velocity analysis:

$$\underline{\omega}_j = \underline{\omega}_i + \underline{\omega}_{j/i} = \underline{\omega}_i + \underline{\underline{A}}_i \underline{\omega}^{(i)}_{j/i}$$

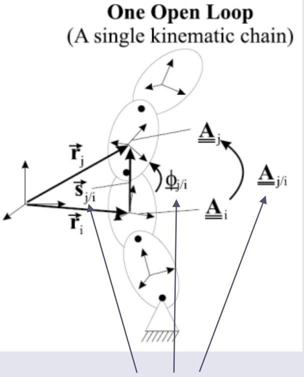
$$\dot{\underline{r}}_j = \dot{\underline{r}}_i + \dot{\underline{s}}_{j/i} = \dot{\underline{r}}_i + \underline{\underline{\tilde{\omega}}}_i \underline{\underline{A}}_i \underline{s}^{(i)}_{j/i} + \underline{\underline{A}}_i \dot{\underline{s}}^{(i)}_{j/i}$$

Acceleration analysis:

$$\dot{\underline{\omega}}_j = \dot{\underline{\omega}}_i + \underline{\underline{A}}_i \dot{\underline{\omega}}^{(i)}_{j/i} + \underline{\underline{\tilde{\omega}}}_i \underline{\underline{A}}_i \underline{\omega}^{(i)}_{j/i}$$

$$\ddot{\underline{r}}_j = \ddot{\underline{r}}_i + \ddot{\underline{s}}_{j/i}$$

$$= \ddot{\underline{r}}_i + \dot{\underline{\underline{\tilde{\omega}}}}_i \underline{\underline{A}}_i \underline{s}^{(i)}_{j/i} + \underline{\underline{\tilde{\omega}}}_i \underline{\underline{\tilde{\omega}}}_i \underline{\underline{A}}_i \underline{s}^{(i)}_{j/i} + 2\underline{\underline{\tilde{\omega}}}_i \underline{\underline{A}}_i \dot{\underline{s}}^{(i)}_{j/i} + \underline{\underline{A}}_i \ddot{\underline{s}}^{(i)}_{j/i}$$



Relative translation, joint coordinates and transformation matrix.

Closed-loop systems

Closed-loop systems

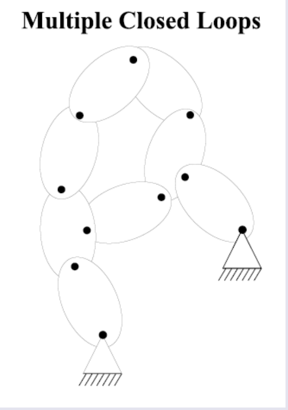
Closed-loop systems are more complicated. Typically, we get nonlinear equations, which cannot be handled analytically. We shall formulate this in a general manner that allows us to solve the equations numerically in a systematical way.

Method of appended driving constraints

- Assemble the constraint equations, including drivers, into one system of nonlinear equations.
- Position analysis:

$$\underline{\Phi}(\underline{q}(t),t)=0$$

where: $\underline{q}(t)$ are the system coordinates.
t is time.

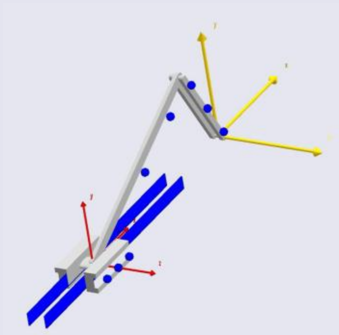


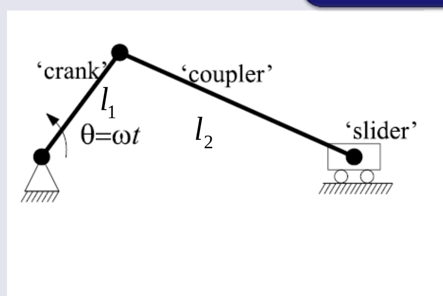
Slider-crank example

Set up kinematic constraint equations for the slider-crank mechanism using the method of appended driving constraints.

Blackboard

The kinematic driver equation is given as: $\theta = \omega t$





M-Tech

3D constraint equations

- Motion of a point in 3D space.
- Constraint equation
  - Spherical join
  - Two perpend
  - Two parallel
  - Universal joi
  - Revolute joi
  - Spherical-sp
  - Driver const

But first a break!

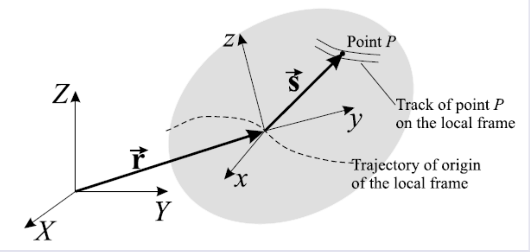
M-Tech

3D constraint equations

- Motion of a point in 3D space.
- Constraint equations:
  - Spherical joint.
  - Two perpendicular vectors.
  - Two parallel vectors.
  - Universal joint.
  - Revolute joint.
  - Spherical-spherical joint.
  - Driver constraints.

M-Tech

Motion of a point in 3D space



Global coordinates of the point P:

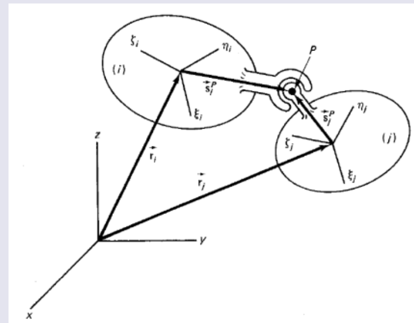
$$\underline{r}^{(P)} = \underline{r} + \underline{s} = \underline{r} + \underline{\underline{A}}\,\underline{s}'$$

M-Tech

Spherical joint constraints

Position constraint:

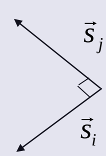
$$\underline{\Phi}^{(s,3)}(\underline{q},t)=\underline{r}_i+\underline{\underline{A}}_i\,\underline{s}_i^{'P}-\left(\underline{r}_j+\underline{\underline{A}}_j\,\underline{s}_j^{'P}\right)=0$$



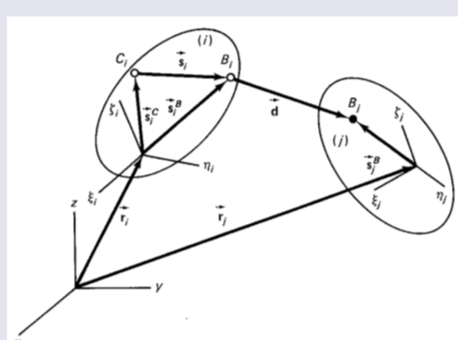
Two perpendicular vectors (type I)

Position constraint:

$$\underline{\Phi}^{(n1,1)}=\underline{s}_i^{T}\,\underline{s}_j=0$$



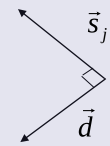
The two vectors, $\underline{s}_i^{'}$ and $\underline{s}_j^{'}$ have constant positions relative to the bodies.



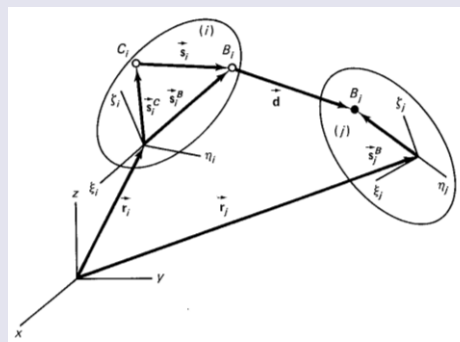
Two perpendicular vectors (type II)

Position constraint:

$$\underline{\Phi}^{(n2,1)} = \underline{s}_i^{T}\,\underline{d} = 0$$



$\underline{s}_i^{'}$ has constant position relative to body *i* whereas $\underline{d}$ is defined between two points on the bodies.



Two parallel vectors

Position constraint:

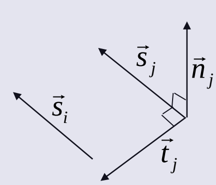
$$\underline{\Phi}^{(p,2)} = \underline{s}_i \times \underline{s}_j = \underline{\underline{\tilde{s}}}_i\,\underline{s}_j = 0$$

NB! This provides 3 constraints rather than two, so one of the three equations must be obmitted.

Alternatively, express two perpendicular vectors:

$$\underline{\Phi}^{(p,2)} = \begin{bmatrix} \underline{s}_i^{T}\,\underline{t}_j \\ \underline{s}_i^{T}\,\underline{n}_j \end{bmatrix} = 0$$

These constraints are always valid, but require the definition of two new vectors.

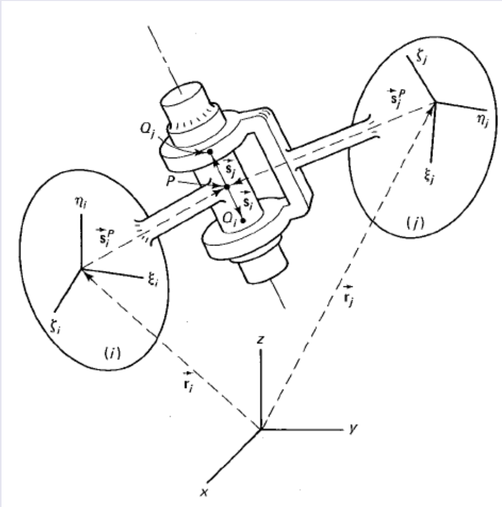


Revolute joint constraints

Position constraint:

$$\underline{\Phi}^{(s,3)} = 0$$

$$\underline{\Phi}^{(p,2)} = 0$$

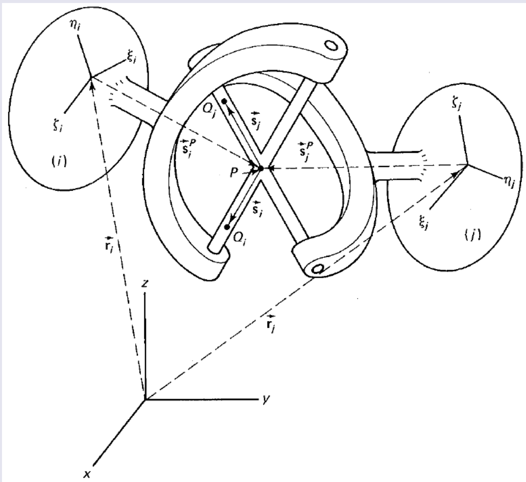


Universal joint constraints

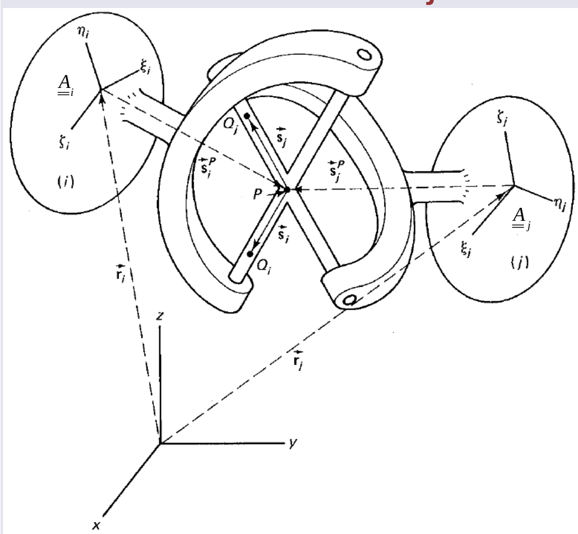
Position constraint:

$$\underline{\Phi}^{(s,3)} = 0$$

$$\underline{\Phi}^{(n1,1)} = 0$$



Universal joint constraints



$$\underline{s}_i = \underline{\underline{A}}_i \underline{s}_i'$$

$$\underline{s}_j = \underline{\underline{A}}_j \underline{s}_j'$$

Position constraint:

$$\underline{\Phi}^{(s,3)} = \underline{r}_i + \underline{\underline{A}}_i \underline{s}_i^{'P} - \left(\underline{r}_j + \underline{\underline{A}}_j \underline{s}_j^{'P}\right) = 0$$

$$\underline{\Phi}^{(n1,1)} = \underline{s}_i^T \underline{s}_j = 0$$

$$= \left(\underline{\underline{A}}_i \underline{s}_i'\right)^T \left(\underline{\underline{A}}_j \underline{s}_j'\right) = 0$$

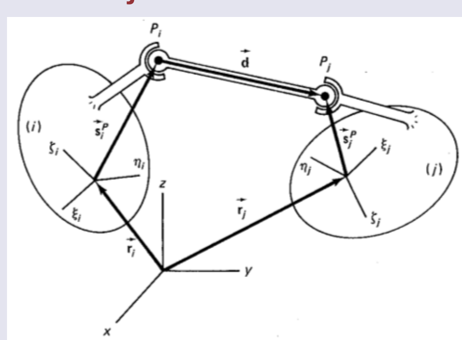
Spherical-spherical joint constraint

Position constraint:

$$\underline{\Phi}^{(s-s,1)} = \underline{d}^T \underline{d} - l^2 = 0,$$

$$\underline{d} = \underline{r}_j + \underline{\underline{A}}_j s_j^{'P} - \underline{r}_i + \underline{\underline{A}}_i s_i^{'P}$$

Many more joint constraints can be found in Nikravesh Section 7.2.



Driver constraints

Used to impose motion on the model.

There must be as many driver constraints as model DOF for a kinematically determinate system.

Position constraint:

$$\underline{\Phi}^{(d)}(\underline{q},t)=0$$

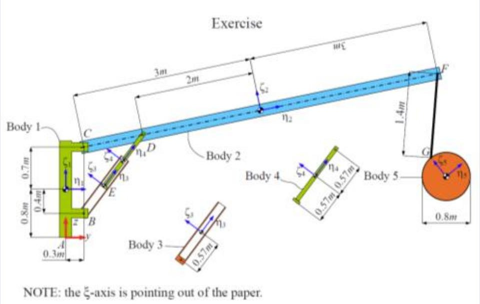
Example:

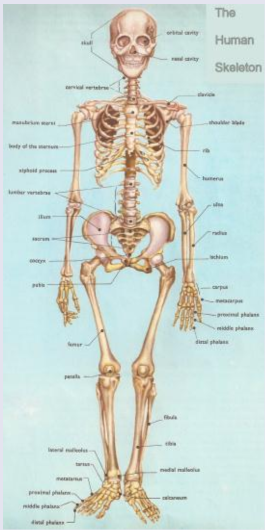
$$\underline{\Phi}^{(d)}(\underline{q},t)=\Psi(\underline{q})-\Gamma(t)=0$$

Some property of the model is explicitly specified by the kinematic driver. E.g. a joint angle

Exercise

Work on **part 1** of the exercises.



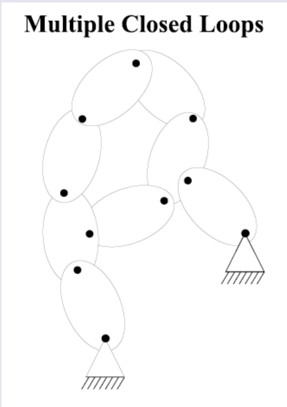


Velocity and acceleration analysis

Method of appended driving constraints

- Position analysis:

$$\underline{\Phi}(\underline{q}(t), t) = 0$$



Method of appended driving constraints

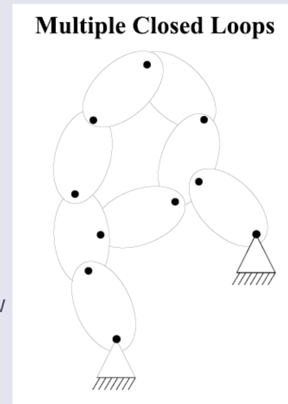
- Velocity analysis:

$$\dot{\underline{\Phi}}(\underline{q},\dot{\underline{q}},t) = \underline{\Phi}_q \dot{\underline{q}} + \underline{\Phi}_t = 0$$

- The jacobian matrix and $\underline{\Phi}_t$:

$$\underline{\Phi}_q = \begin{bmatrix} & \vdots & \\ \cdots & \dfrac{\partial \Phi_i}{\partial q_j} & \cdots \\ & \vdots & \end{bmatrix} \qquad \underline{\Phi}_t = \begin{bmatrix} \vdots \\ \dfrac{\partial \Phi_i}{\partial t} \\ \vdots \end{bmatrix} \leftarrow \text{i'th row}$$

↑ j'th column

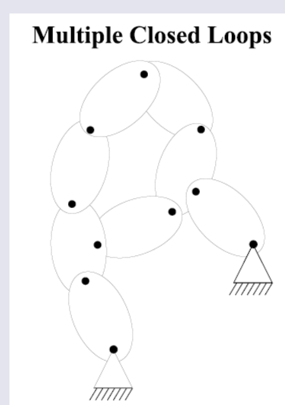


Method of appended driving constraints

Acceleration analysis:

$$\ddot{\underline{\Phi}}(\underline{q},\dot{\underline{q}},\ddot{\underline{q}},t) = \underline{\Phi}_q \ddot{\underline{q}} + \left(\underline{\Phi}_q \dot{\underline{q}}\right)_q \dot{\underline{q}} + 2\underline{\Phi}_{qt} \dot{\underline{q}} + \underline{\Phi}_{tt} = 0$$

$$\underline{\Phi}_q \ddot{\underline{q}} = \underbrace{-\left(\underline{\Phi}_q \dot{\underline{q}}\right)_q \dot{\underline{q}} - 2\underline{\Phi}_{qt} \dot{\underline{q}} - \underline{\Phi}_{tt}}_{\gamma(q,\dot{q},t)}$$

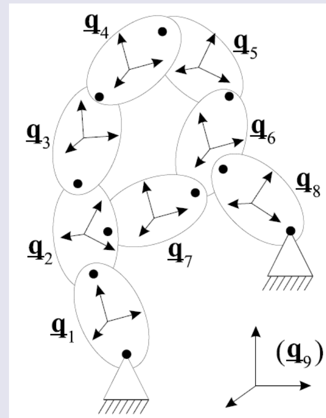


Different position, velocity and acceleration coordinates

It is sometimes desirable to use different coordinates for position analysis than for velocity and acceleration analysis.

For instance with cartesian coordinates:

$$\underline{q}_i = [\underline{r}_i \quad ??] \qquad \underline{v}_i = [\dot{\underline{r}}_i \quad \underline{\omega}_i'] \qquad \dot{\underline{v}}_i = [\ddot{\underline{r}}_i \quad \dot{\underline{\omega}}_i']$$



Velocity analysis:

$$\dot{\Phi}(\underline{q},\underline{v},t) = \Phi_{\hat{q}}\underline{v} + \Phi_t = 0$$

Acceleration analysis:

$$\ddot{\Phi}(\underline{q},\underline{v},\dot{\underline{v}},t) = \Phi_{\hat{q}}\dot{\underline{v}} + \left(\Phi_{\hat{q}}\underline{v}\right)_{\hat{q}}\underline{v} + 2\Phi_{\hat{q}t}\underline{v} + \Phi_{tt} = 0$$

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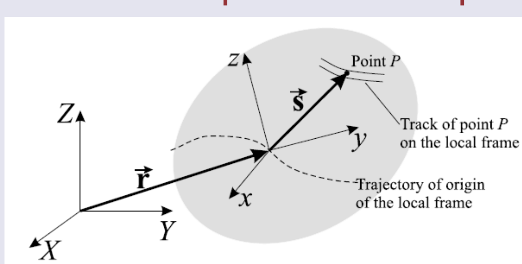
Motion of a point in 3D space



Global coordinates of the point P:

$$\underline{r}^{(P)} = \underline{r} + \underline{s} = \underline{r} + \underline{\underline{A}}\underline{s}'$$

Velocity of the point P:

$$\dot{\underline{r}}^{(P)} = \dot{\underline{r}} + \dot{\underline{\underline{A}}}\underline{s}' + \underline{\underline{A}}\dot{\underline{s}}' = \dot{\underline{r}} + \underline{\underline{A}}\,\tilde{\underline{\underline{\omega}}}'\underline{s}' + \cancel{\underline{\underline{A}}\dot{\underline{s}}'}$$

Acceleration of the point P:

$$\ddot{\underline{r}}^{(P)} = \ddot{\underline{r}} + \underline{\underline{A}}\,\tilde{\underline{\underline{\omega}}}'\tilde{\underline{\underline{\omega}}}'\underline{s}' + \underline{\underline{A}}\,\dot{\tilde{\underline{\underline{\omega}}}}'\underline{s}' + 2\cancel{\underline{\underline{A}}\,\tilde{\underline{\underline{\omega}}}'}\dot{\underline{s}}' + \cancel{\underline{\underline{A}}\ddot{\underline{s}}'}$$

Zero for rigid bodies

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Spherical joint constraints

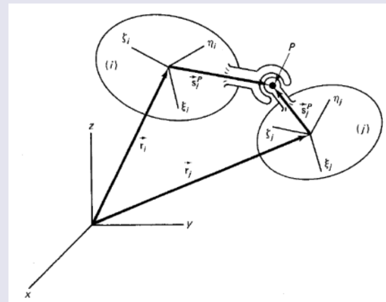
Position constraint:

$$\underline{\Phi}^{(s,3)}(\underline{q},t)=\underline{r}_i+\underline{\underline{A}}_i\underline{s}_i^{'P}-\left(\underline{r}_j+\underline{\underline{A}}_j\underline{s}_j^{'P}\right)=0$$

Velocity constraint:

$$\dot{\underline{\Phi}}^{(s,3)}(\underline{q},\underline{v},t)=\dot{\underline{r}}_i+\underline{\underline{A}}_i\underline{\underline{\tilde{\omega}}}_i^{'}\underline{s}_i^{'P}-\left(\dot{\underline{r}}_j+\underline{\underline{A}}_j\underline{\underline{\tilde{\omega}}}_j^{'}\underline{s}_j^{'P}\right)=0$$

$$=\underbrace{\left[I \quad -\underline{\underline{A}}_i\underline{s}_i^{'P}\right]}_{\underline{\underline{\Phi}}_{\underline{v}_i}}\underline{v}_i-\underbrace{\left[I \quad -\underline{\underline{A}}_j\underline{s}_j^{'P}\right]}_{\underline{\underline{\Phi}}_{\underline{v}_j}}\underline{v}_j=0$$



Acceleration constraint:

$$\ddot{\underline{\Phi}}^{(s,3)}(\underline{q},\underline{v},\dot{\underline{v}},t)=\left[I \quad -\underline{\underline{A}}_i\underline{\tilde{s}}_i^{'P}\right]\dot{\underline{v}}_i-\left[I \quad -\underline{\underline{A}}_j\underline{\tilde{s}}_j^{'P}\right]\dot{\underline{v}}_j-\underbrace{\underline{\underline{\tilde{\omega}}}_j\underline{\underline{\tilde{\omega}}}_j\underline{s}_j^{P}+\underline{\underline{\tilde{\omega}}}_i\underline{\underline{\tilde{\omega}}}_i\underline{s}_i^{P}}_{\gamma(q,v,\dot{v},t)}=0$$

Two perpendicular vectors (type I)

Position constraint:

$$\underline{\Phi}^{(n1,1)}=\underline{s}_i^T\underline{s}_j=\left(\underline{\underline{A}}_i\underline{s}_i^{'}\right)^T\left(\underline{\underline{A}}_j\underline{s}_j^{'}\right)=\underline{s}_i^{'T}\underline{\underline{A}}_i^T\underline{\underline{A}}_j\underline{s}_j^{'}=0$$

Velocity constraint:

$$\dot{\underline{\Phi}}^{(n1,1)}=\underline{s}_i^{'T}\left(\underline{\underline{A}}_i\underline{\underline{\tilde{\omega}}}_i^{'}\right)^T\underline{\underline{A}}_j\underline{s}_j^{'}+\underline{s}_i^{'T}\underline{\underline{A}}_i^T\underline{\underline{A}}_j\underline{\underline{\tilde{\omega}}}_j^{'}\underline{s}_j^{'}$$

$$=\underbrace{-\underline{s}_j^{'T}\underline{\underline{A}}_j^T\underline{\underline{A}}_i\underline{\underline{\tilde{s}}}_i^{'}}_{\underline{\underline{\Phi}}_{\underline{\omega}_i^{'}}}\underline{\omega}_i^{'}\underbrace{-\underline{s}_i^{'T}\underline{\underline{A}}_i^T\underline{\underline{A}}_j\underline{\underline{\tilde{s}}}_j^{'}}_{\underline{\underline{\Phi}}_{\underline{\omega}_j^{'}}}\underline{\omega}_j^{'}=0$$

Hint: $\left(\tilde{a}b=-\tilde{b}a\right)$

Acceleration constraint:

$$\ddot{\underline{\Phi}}^{(n1,1)}=-\underline{s}_j^{'T}\underline{\underline{A}}_j^T\underline{\underline{A}}_i\underline{\underline{\tilde{s}}}_i^{'}\underline{\omega}_i^{'}-\underline{s}_i^{'T}\underline{\underline{A}}_i^T\underline{\underline{A}}_j\underline{\underline{\tilde{s}}}_j^{'}\underline{\omega}_j^{'}=0$$

$$=-\underline{s}_j^{'T}\underline{\underline{A}}_j^T\underline{\underline{A}}_i\underline{\underline{\tilde{s}}}_i^{'}\dot{\underline{\omega}}_i^{'}-\underline{s}_i^{'T}\underline{\underline{A}}_i^T\underline{\underline{A}}_j\underline{\underline{\tilde{s}}}_j^{'}\dot{\underline{\omega}}_j^{'}-\underbrace{\underline{s}_j^{'T}\underline{\underline{A}}_j^T\underline{\underline{A}}_i\underline{\underline{\tilde{\omega}}}_i^{'}\underline{\underline{\tilde{s}}}_i^{'}\underline{\omega}_i^{'}-2\underline{s}_i^{'T}\left(\underline{\underline{A}}_i\underline{\underline{\tilde{\omega}}}_i^{'}\right)^T\underline{\underline{A}}_j\underline{\underline{\tilde{s}}}_j^{'}\underline{\omega}_j^{'}-\underline{s}_i^{'T}\underline{\underline{A}}_i^T\underline{\underline{A}}_j\underline{\omega}_j^{'}\underline{\underline{\tilde{s}}}_j^{'}\underline{\omega}_j^{'}}_{\gamma(q,v,\dot{v})}=0$$

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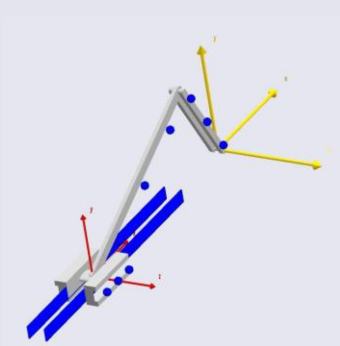
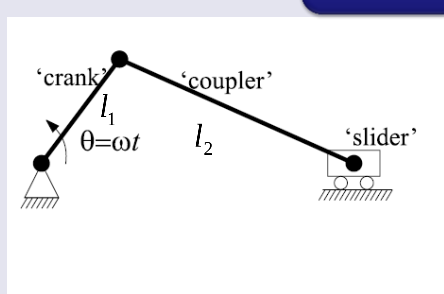
The procedure is the same for the remaining constraint types.

Slider-crank example

Derive the Jacobian matrix and gamma vector for the slider-crank mechanism.

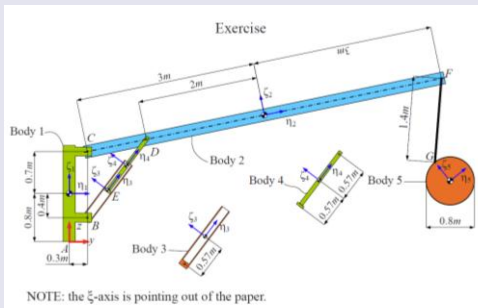
Blackboard

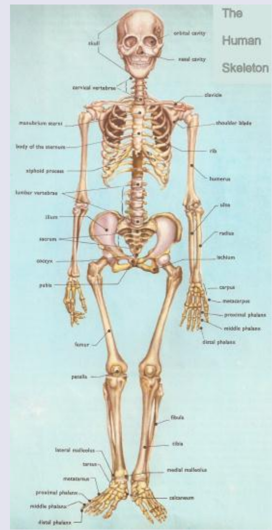




Exercise

Work on **part 2** of the exercises.





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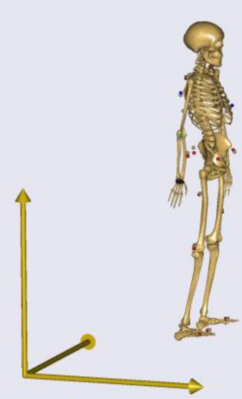
Thanks for your attention!



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