PhD Course on

Advanced Mechanics of Mechanical Systems

Day 2: Dynamics

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Sept 18-21, 2012

Part I: Multiple-rigid-body Dynamics

- Newton-Euler Equation
- Lagrange Equation
- Numerical implementations
- Formulations for improved performance
- Other considerations

Examples of dynamic modeling

- Dynamic modeling of Hydrama 922D dump truck

Dynamic modeling of HMF 2420-K5 Knuckle Boom Loader

Fundamentals: Newton-Euler equations

- Translation equation for a body

$$\boxed{m\ddot{\mathbf{r}} = \mathbf{f}} \quad \text{with} \quad \mathbf{f} = \sum_{i=1}^{p} \mathbf{f}_i \qquad (1.1)$$

- Rotational equation (Euler equation)

$$\boxed{\mathbf{n} = \mathbf{J}\dot{\omega} + \tilde{\omega}\mathbf{J}\omega} \qquad (1.2)$$

Remarks:

- The linear acceleration by Newton 2nd law is given for center of mass. The linear acceleration changes within the rigid body.
- The linear acceleration is NOT dependent to linear velocity
- The angular acceleration IS dependent to the angular velocity
- The moment in Euler equation is taken about center of mass. NOT other points in the fixed frame.
  - In some cases, Euler equation can also be described in term of local components, i.e.,

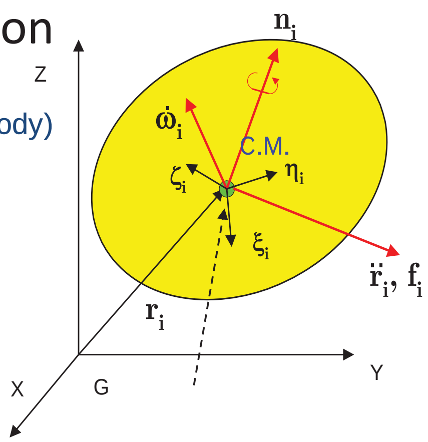
    $$\mathbf{n}' = \mathbf{J}'\dot{\omega}' + \tilde{\omega}'\mathbf{J}'\omega'$$

  - In this way, J' is a constant. (1.3)

Newton-Euler equation

Motion equation of multiple unconstrained bodies (free body)

For a single body in a multibody system, its EOM is:



$$\begin{aligned} m_i\ddot{\mathbf{r}}_i &= \mathbf{f}_i \\ \mathbf{J}_i\dot{\boldsymbol{\omega}}_i + \tilde{\omega}_i\mathbf{J}_i\boldsymbol{\omega}_i &= \mathbf{n}_i \end{aligned}$$

In matrix form

$$\begin{bmatrix} m_i\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_i \end{bmatrix}\begin{bmatrix} \ddot{\mathbf{r}}_i \\ \dot{\boldsymbol{\omega}}_i \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \tilde{\omega}_i\mathbf{J}_i\boldsymbol{\omega}_i \end{bmatrix} = \begin{bmatrix} \mathbf{f}_i \\ \mathbf{n}_i \end{bmatrix}$$

or

$$\mathbf{M}_i\dot{\mathbf{v}}_i + \mathbf{b}_i = \mathbf{g}_i \qquad (1.4)$$

Note: 1. $\ddot{\mathbf{r}}$ and **f** are in the same direction, but $\dot{\omega}$ and **n** are generally not

2. All local coordinates are established at their centers of mass

Newton-Eular equations

- Assembly together the EOMs for all bodies

$$\begin{bmatrix} \mathbf{M}_1 & & & \\ & \mathbf{M}_2 & & \\ & & \cdots & \\ & & & \mathbf{M}_n \end{bmatrix}\begin{bmatrix} \ddot{\mathbf{r}}_1 \\ \dot{\boldsymbol{\omega}}_1 \\ \vdots \\ \ddot{\mathbf{r}}_n \\ \dot{\boldsymbol{\omega}}_n \end{bmatrix} + \begin{bmatrix} 0 \\ \tilde{\boldsymbol{\omega}}_1\mathbf{J}_1\boldsymbol{\omega}_1 \\ \vdots \\ 0 \\ \tilde{\boldsymbol{\omega}}_n\mathbf{J}_n\boldsymbol{\omega}_n \end{bmatrix} = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{n}_1 \\ \vdots \\ \mathbf{f}_n \\ \mathbf{n}_n \end{bmatrix} \qquad (1.5)$$

- In short form $\mathbf{M}\dot{\mathbf{v}} + \mathbf{b} = \mathbf{g}$ with $\mathbf{v} = \begin{bmatrix} \ddot{\mathbf{r}}_1 \\ \dot{\boldsymbol{\omega}}_1 \\ \vdots \\ \ddot{\mathbf{r}}_n \\ \dot{\boldsymbol{\omega}}_n \end{bmatrix}$;

- The equation can be used in inverse dynamics, i.e., to find forces needed for a desired motion.

Dynamic analysis of a system: Lagrange multiplier approach

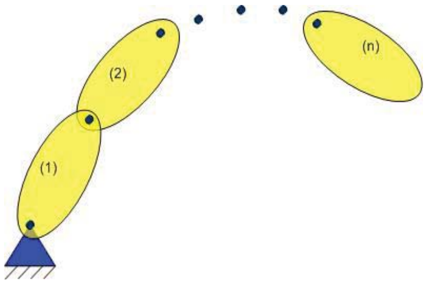
- Eq. (1.3) is established on the basis of unconstrained bodies. In another word, we have to develop freebody diagrams to find out the reaction forces at each joint. Alternatively, we can make use of Lagrange multipliers to include the constraint forces (internal) in the equation.

$$\mathbf{M}_1\dot{\mathbf{v}}_1 + \mathbf{b}_1 = \mathbf{g}_1$$

$$\mathbf{M}_2\dot{\mathbf{v}}_2 + \mathbf{b}_2 = \mathbf{g}_2$$

$$\vdots$$

$$\mathbf{M}_n\dot{\mathbf{v}}_n + \mathbf{b}_n = \mathbf{g}_n$$



and $\mathbf{g}_i = \mathbf{g}_i^{ext} + \mathbf{g}_i^{c}, \quad i = 1, ..., n$

Assume there are m kinematic constraint equations, let

$\boldsymbol{\lambda} = [\lambda_1, \lambda_2, ..., \lambda_m]^T$ Lagrange multiplier

In eq. (1.5), the internal forces are not included explicitly.

Then $\mathbf{g}_i^c = \mathbf{D}_i^T \boldsymbol{\lambda}$

Dynamic equation with Lagrange multipliers:

$$\mathbf{M}\dot{\mathbf{v}} + \mathbf{b} - \mathbf{D}^T\boldsymbol{\lambda} = \mathbf{g}^{ext} \qquad (1.5)$$

Newton-Euler equation

- Eq. (1.5) stands for (n+m) unknowns: the *n* accelerations and *m* Lagrange multipliers. In order to have sufficient number of equations, we have to supply *m* more equations. The acceleration equation can thus be used.
- Assembly with acceleration equation:

$$\mathbf{\Phi}(\mathbf{c}) = \mathbf{0}$$

$$\dot{\mathbf{\Phi}} \equiv \mathbf{D}\mathbf{v} = \mathbf{0}$$

$$\rightarrow \quad \ddot{\mathbf{\Phi}} \equiv \mathbf{D}\dot{\mathbf{v}} + \dot{\mathbf{D}}\mathbf{v} = \mathbf{0}$$

We have $\mathbf{M}\dot{\mathbf{v}} + \mathbf{b} - \mathbf{D}^T\boldsymbol{\lambda} = \mathbf{g}^{ext}$

$$\mathbf{D}\dot{\mathbf{v}} = \boldsymbol{\gamma}; \qquad \boldsymbol{\gamma} = -\dot{\mathbf{D}}\mathbf{v}$$

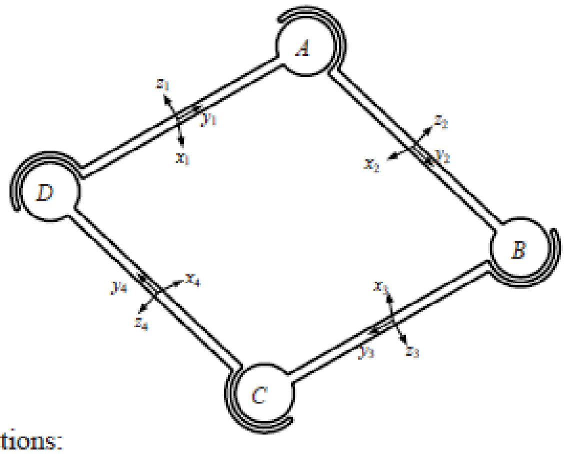
In matrix form

$$\begin{bmatrix} \mathbf{M} & -\mathbf{D}^T \\ \mathbf{D} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{v}} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{g}^{ext} - \mathbf{b} \\ \boldsymbol{\gamma} \end{bmatrix} \qquad (1.6)$$

Now the equation can be readily solved numerically.

Example 1

- A multibody system with four spherical joints



Constraint equations:

$$\Phi = \begin{bmatrix} \mathbf{r}_1 + \mathbf{A}_1\mathbf{s}'_{1A} - (\mathbf{r}_2 + \mathbf{A}_2\mathbf{s}'_{2A}) \\ \mathbf{r}_2 + \mathbf{A}_2\mathbf{s}'_{2B} - (\mathbf{r}_3 + \mathbf{A}_3\mathbf{s}'_{3B}) \\ \mathbf{r}_3 + \mathbf{A}_3\mathbf{s}'_{3C} - (\mathbf{r}_4 + \mathbf{A}_4\mathbf{s}'_{4C}) \\ -(\mathbf{r}_1 + \mathbf{A}_1\mathbf{s}'_{1D}) + \mathbf{r}_4 + \mathbf{A}_4\mathbf{s}'_{4D} \end{bmatrix} = \mathbf{0}$$

Example 1 (cont'd)

- Velocity and acceleration equations, taking the first constraint as an example

$$\dot{\mathbf{\Phi}}_1 = \dot{\mathbf{r}}_1 - \tilde{\mathbf{s}}_{1A}\boldsymbol{\omega}_1 - \dot{\mathbf{r}}_2 + \tilde{\mathbf{s}}_{2A}\boldsymbol{\omega}_2$$
$$\ddot{\mathbf{\Phi}}_1 = \ddot{\mathbf{r}}_1 - \tilde{\mathbf{s}}_{1A}\dot{\boldsymbol{\omega}}_1 + \tilde{\boldsymbol{\omega}}_1\tilde{\boldsymbol{\omega}}_1\mathbf{s}_{1A} - \ddot{\mathbf{r}}_2 + \tilde{\mathbf{s}}_{2A}\boldsymbol{\omega}_2 - \tilde{\boldsymbol{\omega}}_2\tilde{\boldsymbol{\omega}}_2\mathbf{s}_{2A}$$

Make use of

$$\mathbf{r}_i^P = \mathbf{r}_i + \mathbf{s}_i^P = \mathbf{r}_i + \mathbf{A}_i\,\mathbf{s}_i'^P$$
$$\dot{\mathbf{r}}_i^P = \dot{\mathbf{r}}_i + \dot{\mathbf{s}}_i^P = \dot{\mathbf{r}}_i - \tilde{\mathbf{s}}_i^P\boldsymbol{\omega}_i$$
$$\ddot{\mathbf{r}}_i^P = \ddot{\mathbf{r}}_i + \ddot{\mathbf{s}}_i^P = \ddot{\mathbf{r}}_i - \tilde{\mathbf{s}}_i^P\dot{\boldsymbol{\omega}}_i + \tilde{\boldsymbol{\omega}}_i\tilde{\boldsymbol{\omega}}_i\mathbf{s}_i^P$$

Matrix D of the system

$$\begin{bmatrix} \mathbf{I} & -\tilde{\mathbf{s}}_{1A} & -\mathbf{I} & \tilde{\mathbf{s}}_{2A} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & -\tilde{\mathbf{s}}_{2B} & -\mathbf{I} & -\tilde{\mathbf{s}}_{3B} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & -\tilde{\mathbf{s}}_{3C} & -\mathbf{I} & \tilde{\mathbf{s}}_{4C} \\ -\mathbf{I} & \tilde{\mathbf{s}}_{1D} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} & -\tilde{\mathbf{s}}_{4D} \end{bmatrix}$$

Lagrange Equation (with independent coordinates)

- Let the vector **q** represents a set of *n* unknown **independent** coordinates

- Let *T* ($\dot{\mathbf{q}}$**, q)** be the kinetic energy of the system**,** *V(***q***)* the potential energy and
- The Lagrangian: *L= T-V*
- $\mathbf{Q}_{ex}(\mathbf{q})$ the vector of generalized external forces (non conservative forces) acting along the dependent coordinates **q** of a constrained mechanical system**.**

- **The Lagrange equation of the first kind** is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial L}{\partial \mathbf{q}} = \mathbf{Q}^{ex} \qquad (1.7)$$

Lagrange Equation (with dependent coordinates)

- Let the vector **q** represents a set of *n* unknown dependent coordinates*, m* is the total number of independent constraint equations (geometric and kinematic), and *f=n-m* is the number of dynamic degrees of freedom (dof). The constraint conditions are written in the following general form:

$$\Phi(\mathbf{q}, t) = \mathbf{0}$$

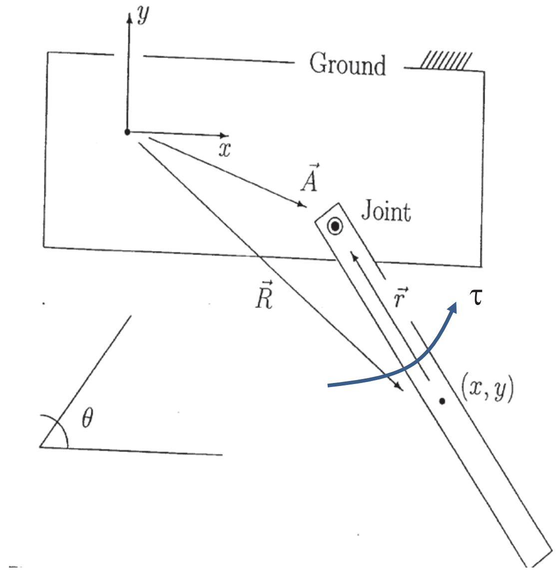
- Let *T* ($\dot{\mathbf{q}}$**, q)** be the kinetic energy of the system**,** *V(***q***)* the potential energy and
- **The Lagrange's equations of the second kind**:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right) - \frac{\partial L}{\partial \mathbf{q}} + \mathbf{\Phi}_{\mathbf{q}}^{\mathrm{T}}\boldsymbol{\lambda} = \mathbf{Q}_{\mathbf{ex}} \qquad (1.8)$$

- The matrix $\mathbf{\Phi}_{\mathbf{q}}$ is the Jacobian matrix of the constraint equations (1.8) . The vector $\boldsymbol{\lambda}$ in (1.8) defines the Lagrange multipliers.

Example 2: A 1-dof pendulum
Formulation the equation of motoin

- New-Euler equation



- Lagrange equation

1. Kinetic enegry $$T = \frac{1}{2}\left(m\dot{x}^2 + m\dot{y}^2 + I\dot{\theta}^2\right)$$
2. Potential energy $$V = mgy$$
3. Lagrangian funciton $$L = T - V$$
4. Lagrange equation $$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} + \Phi_q^{\mathrm{T}}\lambda = Q$$

$$\Downarrow$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = \begin{pmatrix} m\ddot{x} \\ m\ddot{y} \\ I\ddot{\theta} \end{pmatrix} + \begin{pmatrix} 0 \\ mg \\ 0 \end{pmatrix}$$

$$\Downarrow$$

$$\begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & I \end{pmatrix} \begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\theta} \end{pmatrix} + \begin{pmatrix} 0 \\ mg \\ 0 \end{pmatrix} + \Phi_q^{\mathrm{T}}\lambda = Q;$$

Numerical solutions

- While there are many ways availalbe, a convenient method is to convert them into DAE of first order. This implies that, given the first order derivatives of dependent variables at time t, we can use an integrator to find their new values for time t+Δt.

- Algorithm

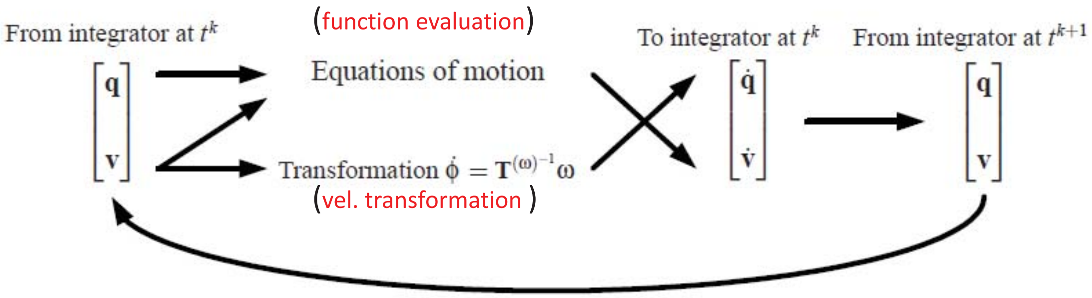
  1. start at a time $t=t_0$ in which the position and velocity are known.

  2. Use eq. (1.6) to solve the accelerations at time $t=t_0$. This step is referred as *function evaluation*

  3. Let $\dot{\mathbf{y}}_t^T = \{\dot{\mathbf{v}}, \dot{\mathbf{q}}\}$, which can be used as input to an integrator of first order differential equations to get $\dot{\mathbf{y}}_{t+\Delta t}^T$

  4. Update time t and go to step 2

Algorithm



- Refer to appendix A (Nikravesh's book) for the transformation between angular rates and angular velocities. For Euler angles of ZXZ convention, the transformation is

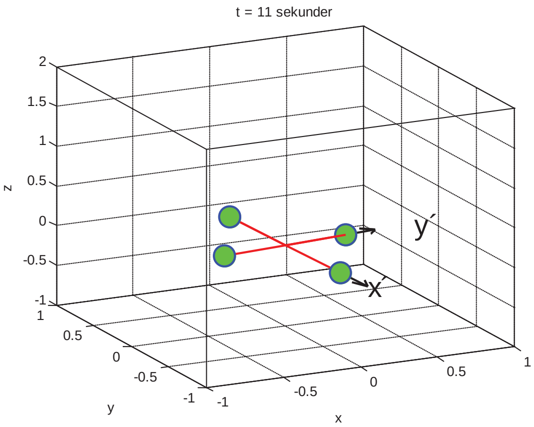
$$\begin{bmatrix} \omega_{(\xi)} \\ \omega_{(\eta)} \\ \omega_{(\zeta)} \end{bmatrix} = \begin{bmatrix} \sin\theta\sin\sigma & \cos\sigma & 0 \\ \sin\theta\cos\sigma & -\sin\sigma & 0 \\ \cos\theta & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\sigma} \end{bmatrix} \qquad (1.9)$$

Example 3. Dyanmic simulation

- A falling-cross (matlab code is available at the course website)

Assumptions:

- Four metal balls connectted rigidly by rods
- The impact of the cross on the ground is characterised with a spring constant and a damping coefficient



New formulation for improved stability

- With the formulation given in eq. (1.6), the problem of numerical stability raises. The reason lies in the integration of diffential equation of the constraints. The general solution of this differential equation is unbounded, thus leading to unstable solutions.

- Look at the second order of differential equation of the constraints

$$\ddot{\mathbf{\Phi}}(\mathbf{q},t) \equiv \mathbf{\Phi}_{\mathbf{q}}\ddot{\mathbf{q}} + \dot{\mathbf{\Phi}}_{\mathbf{q}}\dot{\mathbf{q}} + \dot{\mathbf{\Phi}}_t = \mathbf{0} \qquad (1.10)$$

- Its general solution is in a form of

$$\mathbf{\Phi}(\mathbf{q},t) = \mathbf{a}_1 t + \mathbf{a}_2 \qquad (1.11)$$

  where a_1 and a_2 depend on the initial conditions.

- Solution in the above form is unbounded, making the system unstable.

- A remedy to fix this problem is to mix the system of differential equations with algebraic equations (constraints)

Baumgarte Stabilization

- This is a control theory based method. The idea is to enforce/impose the contraints at the acceleration level by adding the constaint equations in the differential equations as

$$\ddot{\mathbf{\Phi}} + 2\alpha\,\dot{\mathbf{\Phi}} + \beta^2\,\mathbf{\Phi} = \mathbf{0} \tag{1.12}$$

- where α and β are constants

- The solution is now in a form of

$$\mathbf{\Phi} = \mathbf{a}_1\, e^{s_1 t} + \mathbf{a}_2\, e^{s_2 t} \tag{1.13}$$

- with

$$s_1, s_2 = -\,\alpha \pm \sqrt{\alpha^2 - \beta^2} \tag{1.14}$$

- If s_1 and s_2 are negative, the general solution is bounded, thus the stability is guaranteed.

A new formulation

- With the Baumgarte Stabilization method, the dynamic equation of a multibody system can be rewritten as

$$\begin{bmatrix} \mathbf{M} & \mathbf{\Phi}_q^T \\ \mathbf{\Phi}_q & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{q}} \\ \boldsymbol{\lambda} \end{Bmatrix} = \begin{Bmatrix} \mathbf{Q} \\ \mathbf{g} \end{Bmatrix} \tag{1.15}$$

- where

$$\mathbf{g} = -\,\dot{\mathbf{\Phi}}_t - \dot{\mathbf{\Phi}}_q\,\dot{\mathbf{q}} - 2\alpha\left(\mathbf{\Phi}_q\,\dot{\mathbf{q}} + \mathbf{\Phi}_t\right) - \beta^2\,\mathbf{\Phi} \tag{1.16}$$

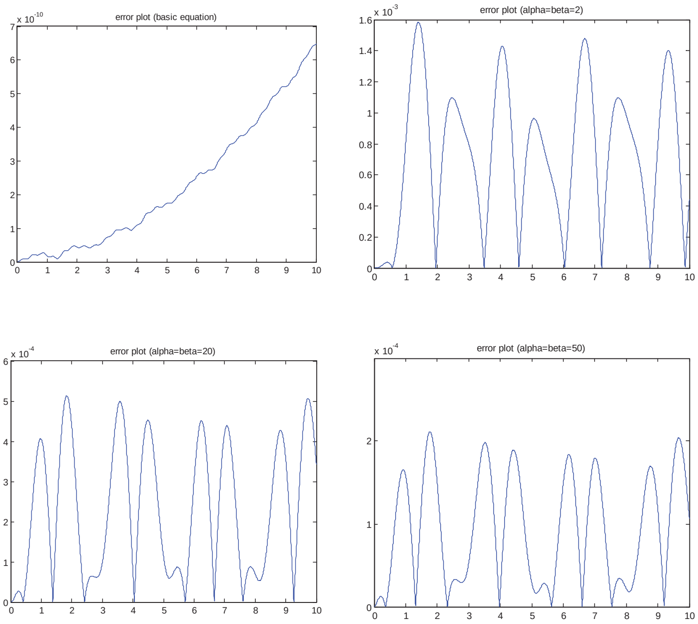
- or

$$\mathbf{g} = \gamma - 2\alpha\dot{\mathbf{\Phi}} - \beta^2\mathbf{\Phi} \tag{1.17}$$

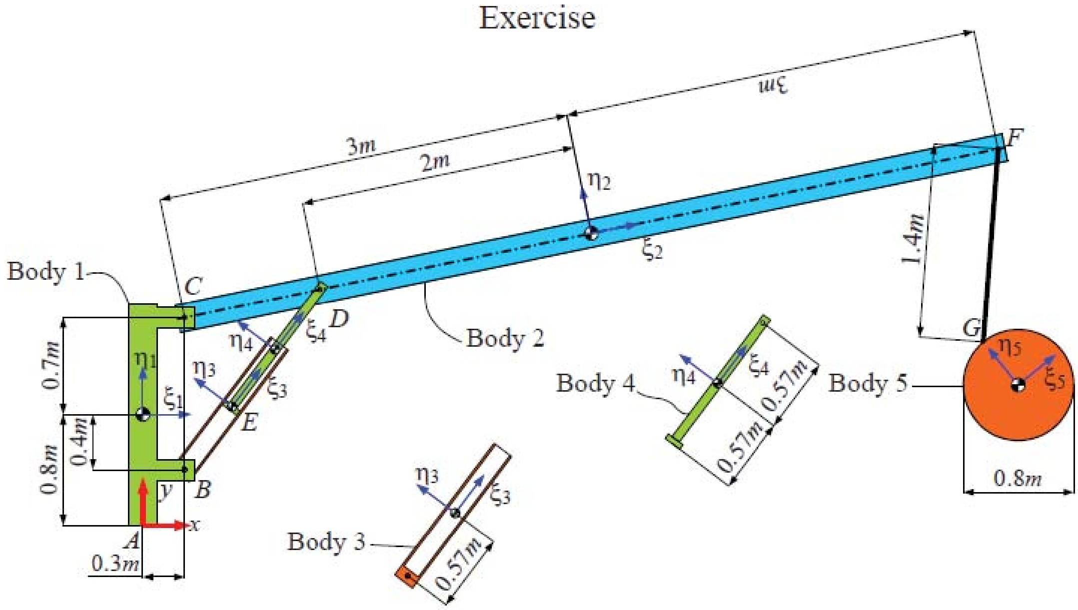
- Remarks:
  - The constants α and β, depending on the problem, are usually set to equal values between 1 to 20.
  - The new formulation is simple---modifying simply the original differential equations, and computational efficient.
  - It has limitations, for example, in dealing with near-singularity configurations.
- Further reading: Chapter 5, J. Jalon and E. Bayo, Kinematic and dynamic simulation of multibody systems.

Example 4: Numerical experiments

- .We use the same example of our exercise (see slide next page)
- The error is defined as the rms values of residuals of all constraint equations.



Exercise



Singularity-free dynamic equation solving

- In the previous formulation, the system may likely reach a singular configuration, when the leading matrix becomes singular.

$$\begin{bmatrix} \mathbf{M} & \mathbf{\Phi}_q^T \\ \mathbf{\Phi}_q & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{q}} \\ \boldsymbol{\lambda} \end{Bmatrix} = \begin{Bmatrix} \mathbf{Q} \\ \mathbf{g} \end{Bmatrix}$$

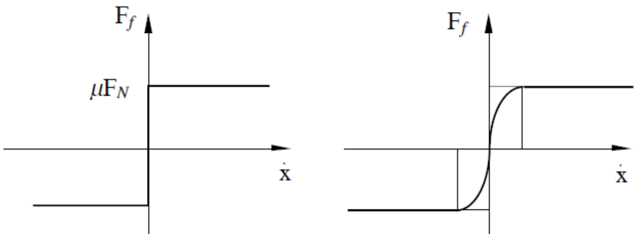
- The consequence of a single position is kinematically a sudden change in the number of degrees of freedom, either lost or increases one or more degrees. In the numerical solving, the accelerations cannot be computed, the dyanmic simulation may either crash or lead to a large error. The near-singular configuration will lead to amplified round-off error and resulting erroneous solutions.

- A simple way to deal with the singularity is to detect the ill-conditioning of the Jacobian and let the integrator step over it.
- New formulations are available for robust solutions. One is the penalty-augmented Lagrangian formulation (ch. 10.6, Jalon's book):

$$(\mathbf{M} + \mathbf{\Phi}_q^T \boldsymbol{\alpha} \mathbf{\Phi}_q)\ddot{\mathbf{q}} = = \mathbf{Q} - \mathbf{\Phi}_q^T \boldsymbol{\alpha} (\dot{\mathbf{\Phi}}_q \dot{\mathbf{q}} + \dot{\mathbf{\Phi}}_t + 2\boldsymbol{\Omega} \boldsymbol{\mu} \dot{\mathbf{\Phi}} + \boldsymbol{\Omega}^2 \mathbf{\Phi}) - \mathbf{\Phi}_q^T \boldsymbol{\lambda}^* \quad (1.18)$$

Friction force (ch. 10.1, Jalon's book)

- Coulomb friction model



- To incooperate with Coulomb friction, an additional item can be added to the equation of motion as

$$\begin{bmatrix} \mathbf{M} & \mathbf{\Phi}_q^T \\ \mathbf{\Phi}_q & 0 \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{q}} \\ \boldsymbol{\lambda} \end{Bmatrix} = \begin{Bmatrix} \mathbf{Q} + \mathbf{Q}_f(\boldsymbol{\lambda}) \\ \mathbf{c} \end{Bmatrix} \quad (1.19)$$

With

$$\mathbf{Q}_f = \mu \cdot \mathbf{u}(\mathbf{q})\, \mathbf{E}(\mathbf{q})\, \mathbf{\Phi}_q^T \boldsymbol{\lambda} \quad (1.20)$$

Where μ is frictional coefficient, **u**(**q**) is a matrix describing joint type and geometry

Impact and collisions

- In certain cases, the dynamic modelling may get involved with impact and collisions. It is normally assumed that the duration of impact is very short and impact force is large, all other remaining forces can thus be neglected. Another assumption is the position of the system does not change during the impact, only velocity changes.
- The equation of motion, dring the impact, is:

$$\begin{bmatrix} \mathbf{M} & \mathbf{\Phi}_{\mathbf{q}}^{T} \\ \mathbf{\Phi}_{\mathbf{q}} & 0 \end{bmatrix} \begin{Bmatrix} \Delta\dot{\mathbf{q}} \\ \boldsymbol{\lambda}_{p} \end{Bmatrix} = \begin{Bmatrix} \mathbf{P}^{i} \\ 0 \end{Bmatrix} \quad (1.21)$$

- Where P^i is the integral effect of the impact force Q^i, i.e.

$$\mathbf{P}^{i} = \int_{t_i^-}^{t_i^+} \mathbf{Q}^{i}\, dt \quad (1.22)$$

Other considerations of dynamic modeling

- Backlash (ch. 10.3)

- In dynamic modeling, the complexity of the model depends on the problem.
- Always start modeling from the basic model and validate.
- Check errors of computation

- The best validation is experiments, when applicable.

Further reading:

- P. E. Nikravesh, AN OVERVIEW OF SEVERAL FORMULATIONS FOR MULTIBODY DYNAMICS, *D. Talab� and T. Roche (eds.), Product Engineering,* 189–226

Part II. Gear Dynamics

Gear transmissions are commonly used mechanical systems. As a traditional subject of research and development, gear dynamics has been extensively studied over centuries. However, the demanding for high speed, large-scale transmissions in applications like high-speed machining and wind turbines, poses new challenges for the design and analysis of gear transmissions.

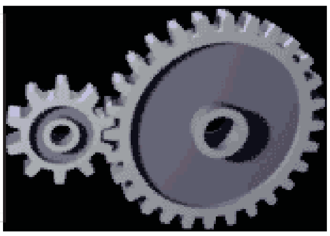
This part of the lecture is aimed at providing an overview of theories for gear dynamics modeling. The following contents are included:

– Gear basics
– Kinematics of gear transmissions
– Rigid body gear dynamics
– Flexible body gear dynamics

1. Gear basics

- Gears are the most common means used for power transmission
- They can be applied between two shafts which are
  - Parallel
  - Collinear
  - Perpendicular and intersecting
  - Perpendicular and nonintersecting
  - Inclined at any arbitrary angle

1. Spur gears
  - Most common used forms for parallel shafts
  - Suitable for low to medium speed application
  - Relatively high ratios can be achieved (< 7)
  - Steel, brass, bronze, cast iron, and plastics



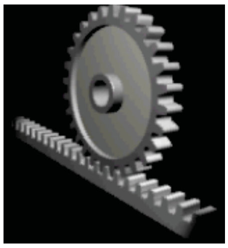
2. Helical gears
  - Teeth are at an angle---helix angle
  - Used for parallel shafts
  - Teeth engage gradually reducing shocks

1. Gear basics

3. Bevel gears
   – They have conical shape
   – Used to change directions
   – Two axes are coplanar

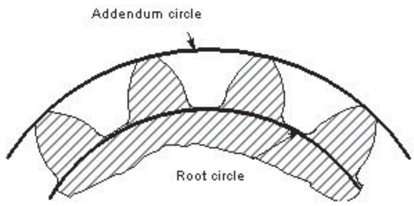
4. Worm gears
   – For large speed reductions: one turn of worm vs one teeth of gear
   – Transmission between two perpendicular and non-intersecting shafts
   – One-way transmission
   – Two axes are not coplanar

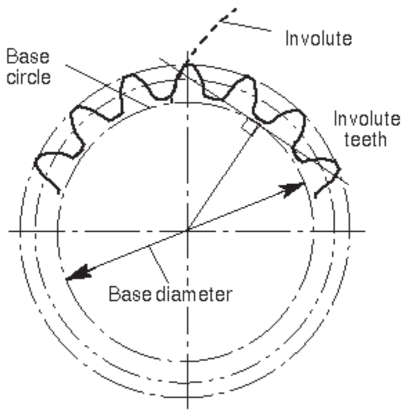
5. Rack and pinion
   – A rack is a gear whose pitch diameter is infinite
   – Used to convert rotary motion to straight line motion



Gear geometry

- Root diameter
  – Diameter of the gear, measured at the base of the tooth.
- Addendum circle (outer circle)
  – the circle that coincides with the tops of the teeth of a gear
- Base circle:
  – a circle of reference of gear teeth profile generation
- pitch circle
  – **An imaginary circle that is centered at gear center an passing the pitch point**
- Addendum and dedendum

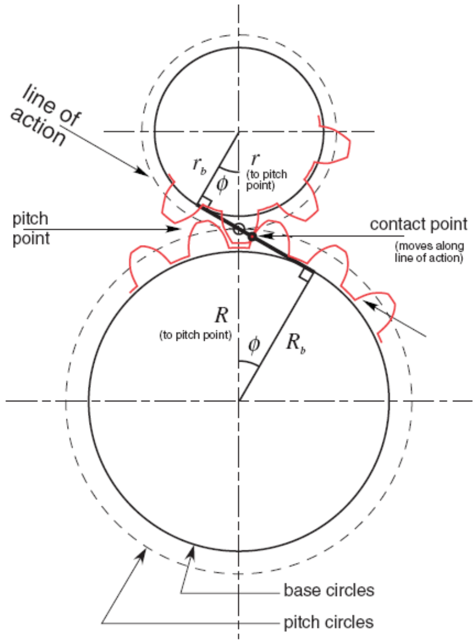




Gear geometry

- Pitch point, p
  - Point where the line of action crosses a line joining the two gear axes.
- Pitch circle, pitch line
  - Circle centered on and perpendicular to the axis and passing through the pitch point.
- Pitch diameter, $d=m.N$
  - Diameter of a pitch circle. Equal to twice the perpendicular distance from the axis to the pitch point (2R). The nominal gear size is usually the pitch diameter.
- Module, m
  - The pitch diameter divided by the number of teeth.
- Pressure angle,

$$\phi = \arccos(r_b / r), \qquad r_b = r\cos\phi$$



Gear geometry

- Line of action
  - A line or curve along which two tooth surfaces are tangent to each other.
  - A line where teeth engagement happens
- Contact ratio
  - Ratio of the arc of action to the circular pitch
  - It stands for the average number of gear tooth pairs in contact on a pair of meshing gears.
- Backlash:
  - The difference between the circle thickness of one gear and the tooth space of the mating gear

2. Gear Kinematics—on the relationships between gear dimension and motion

- Gear ratio (GR): the ratio of number of teeth on driven gear to the number of teeth on driver gear

$$GR = \frac{N_{driven}}{N_{driver}} = \frac{N_2}{N_1} = m \qquad (2.1)$$

- Velocity ratio (VR): the ratio of the angular speed of the driver gear (gear 1) to the angular speed of the driven gear (gear 2)

$$VR = \frac{\omega_{driver}}{\omega_{driven}} = \frac{\omega_1}{\omega_2} \qquad (2.2)$$

- Pitch line velocity: the velocity of the pitch point

$$v_t = r_1\omega_1 = r_2\omega_2 \qquad (2.3)$$

- More expressions of velocity ratio

$$VR = \frac{\omega_{driver}}{\omega_{driven}} = \frac{\omega_1}{\omega_2} = \frac{r_2}{r1} = \frac{d_2}{d1} = \frac{N_2}{N_1} \qquad (2.4)$$

Gear train kinematics

- …. is a series of mating gearsets.
- Train ratio (TR) is the product of all velocity ratios

$$TR = \frac{\omega_{in}}{\omega_{out}} = (VR_1)(VR_2)(VR_3).... \qquad (2.5)$$

Pay attention to the direction of gear rotation

Planet gear trains

- To find the TR, we first fix the arm, then

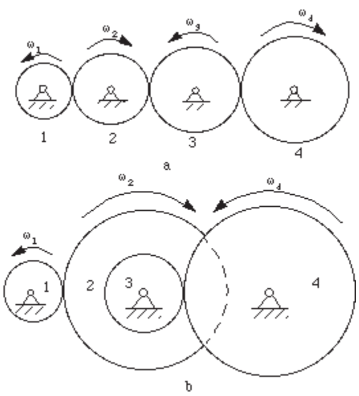
$$TR = (VR_1)(VR_2)(VR_3).... = \frac{\overline{\omega}_{gear1}}{\overline{\omega}_{gearN}} \qquad (2.6)$$

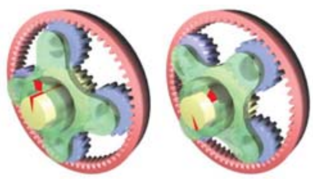
where relative angular velocities are

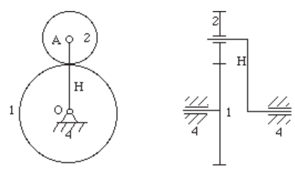
$$\overline{\omega}_{gear1} = \omega_{gear1} - \omega_{arm}, \qquad \overline{\omega}_{gearN} = \omega_{gearN} - \omega_{arm}$$

- Thus

$$(VR_1)(VR_2)(VR_3).... = \frac{\omega_{gear1} - \omega_{arm}}{\omega_{gearN} - \omega_{arm}} \qquad (2.7)$$







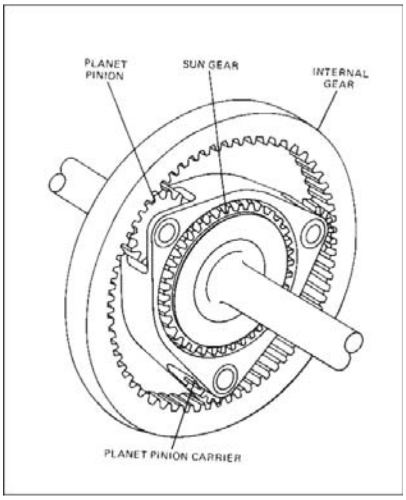
Planet gear train and equivalent 2dof mechanism

Example: Analyzing of planetary gear train

- Problem: determine the train ratio between the sun gear and the arm.
- Given data: The sun has 40 teeth, the planet 20teeth, and the ring gear 80 teeth. The arm is the input and the sun is the output. The ring gear is held stationary.

Solution:

- We look at the gear train from the sun to the ring



- That is $$(-\frac{N_{sun}}{N_{planet}})(\frac{N_{planet}}{N_{ring}}) = \frac{\omega_{ring} - \omega_{arm}}{\omega_{sun} - \omega_{arm}}$$
- or $$(-\frac{40}{20})(\frac{20}{80}) = \frac{0 - \omega_{arm}}{\omega_{sun} - \omega_{arm}}$$
- Hence $$-0.5 = \frac{-1}{\omega_{sun} / \omega_{arm} - 1}$$

$$TR = \omega_{sun} / \omega_{arm} = 3$$

Rigid body gear dynamics

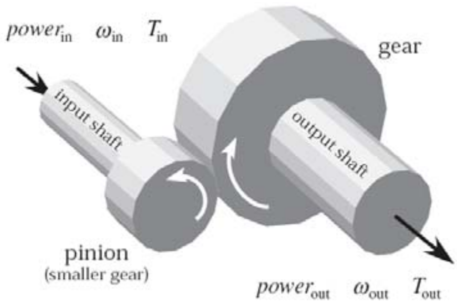
- For a gear transmission shown in the figure

$$power_{in} = T_1\omega_1; \; power_{out} = T_2\omega_2$$

$$power_{in} \approx power_{out}$$

$$T_1\omega_1 = T_2\omega_2 \quad (2.8)$$

$$\frac{T_1}{T_2} = \frac{\omega_2}{\omega_1} = \frac{N_1}{N_2} = n \quad (2.9)$$



- Reflected torque

$$T_{2R} = nT_2 \quad (2.10)$$

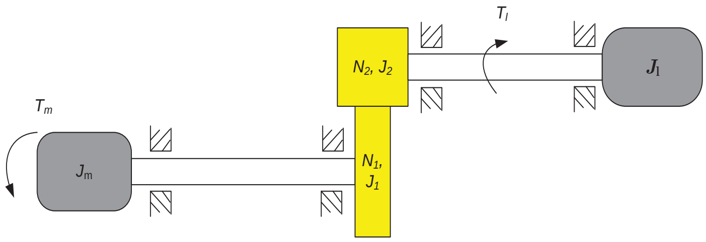
- Reflected moment of inertia: when viewed from the input gear

$$J_{2R} = n^2 J_2 \quad (2.11)$$

- Equation of motion of gear trains

$$T_1 - nT_2 = (J_1 + J_{2R})\alpha \quad (2.12)$$

Example



- Equation of motion of gear train

$$T_m - T_{lR} = (J_m + J_{bR} + J_{lR})\alpha$$

- Where
  - T_m = maximum available motor peak torque
  - J_m = motor rotor inertia
  - J_{bR} = reflected moment of inertia of gearbox
  - T_{lR} = reflected load torque, including frictional torque, for example, T_{lR}= T_{gR}+M_{fR}

- The starting torque (acceleration torque plus steady torque)

$$T_m = T_{lR} + (J_m + J_{bR} + J_{lR})\alpha$$

- The acceleration

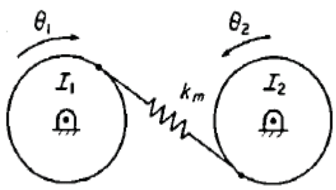
$$\alpha = \frac{T_m - T_{lR}}{J_m + J_{lR} + J_{bR}}$$

- Steady torque

$$T_m = T_{lR}$$

3. Gear dynamics with flexible bodies

Basic nonlinear gear dynamics model



A basic model

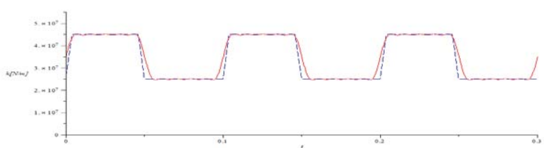
Governing eq. $$m_e\ddot{u} + k_m u = 0 \qquad (3.1)$$

Where:
equivalent mass $$m_e = \frac{I_{1I} I_2}{R_1^2 I_2 + R_2^2 I_1} \qquad (3.2)$$

Transmission error $$u = R_1\theta_1 - R_2\theta_2 \qquad (3.3)$$

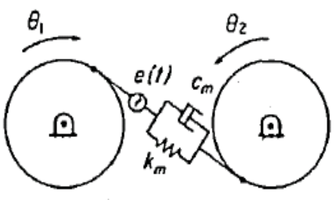
In dyanmic modeling, the following problems need to be sovled

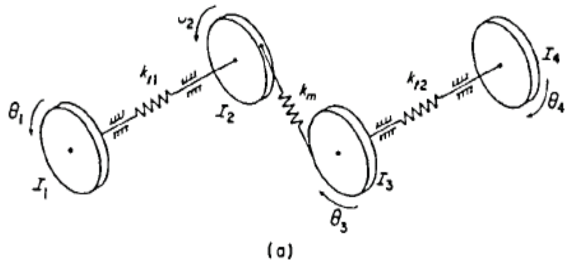
1. Periodic gear meshing force
2. Determination of meshing stiffness
3. Influence of variance such as backlash



Other types of model

$$e_i(t) = E_i \sin(\omega_z t + \alpha)$$





A comprehensive model.

ω_z is the mesh frequency and α is the phase angle. The composite error e_s is the sum of tooth errors of the pinion and gears.

Typical model for a gear train

Mathematical models used in gear dynamics—A review, H. Nevzat Özgüven and D.R. House, Journal of Sound and Vibration , Vol 121, Issue 3, 22 March 1988, Pages 383-411
OPTIMIZATION METHODS FOR SPUR GEAR DYNAMICS, Marco Barbieri, et al., proc. ENOC 2008, Saint Petersburg, Russia, June, 30–July, 4 2008

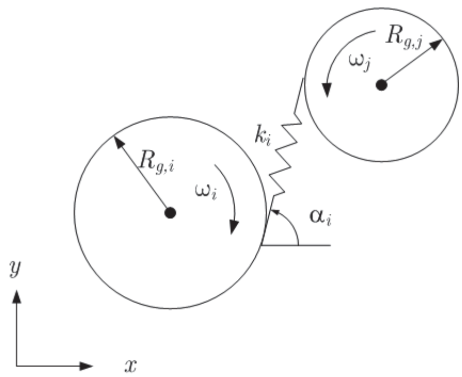
Gear modeling

In dyanmic modeling, the following problems need to be sovled

1. Discontinuous gear meshing force
2. Determination of peroidic meshing stiffness
3. Influence of variance such as backlash

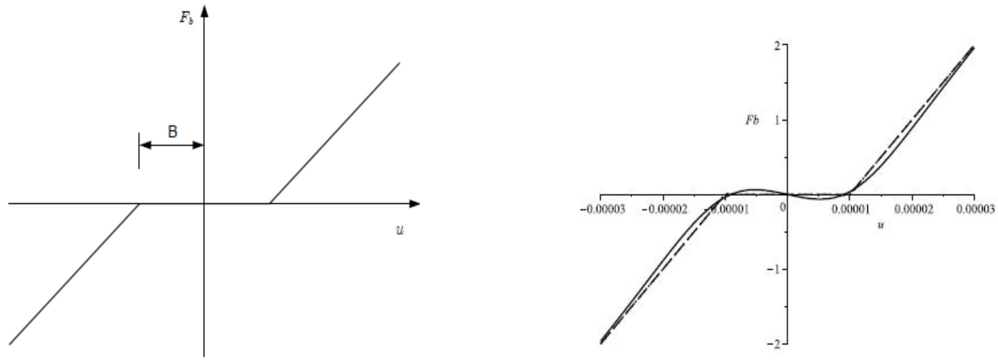
The meshing force calculation has to consider the engagement situations:

1. positive working condition, a situation in which driving gear drives driven gear.
2. negative working condition. In such a situation, a driven gear is in contact with the backside of the driving gear.
3. non engagement.



Model of meshing force with backlash

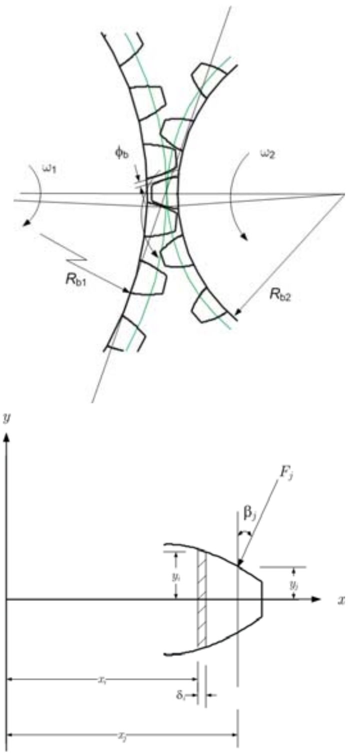
- The meshing force is also subject to backlash(clearance)



(a) (b)

$$F_{b,i} = \begin{cases} k_i(u_i - B) + c_i\dot{u}_i & \text{if } u_i - B > 0 \\ k_i(u_i + B) + c_i\dot{u}_i & \text{if } u_i + B < 0 \\ c_i\dot{u}_i & \text{if } -B < u_i < B \end{cases} \Rightarrow F_{b,i} = k_i[(u_i - B)(1 + tanh(\lambda(u_i - B))) + (u_i + B)(1 - tanh(\lambda(u_i + B)))]/2 + c_i\dot{u}_i$$

Gear meshing stiffness, ref. [4]



- The stiffness at a point j is

$$k_j = \frac{F_j}{y_B + y_L} \qquad (3.4)$$

where y_B is the deformation due to beam effect and y_L is that due to surface contact, which can be calculated by

$$y_L = 4.55(1 - v^2)\frac{F_j}{WE} \qquad y_B = y_B(j) = \sum_{i=1}^{n} y_i(j)$$

With

$$y_i(j) = \frac{F_j}{E_e}(A\cos^2\beta_j + B\cos\beta_j\sin\beta_j + C\sin^2\beta_j)$$

$$A = \frac{1}{3I_j}(\delta_i^3 + 3\delta_i^2 l_i + 3\delta_i l_i^2) + \frac{1}{5S_i}12(1+v)\delta_i$$

$$B = \frac{1}{2I_i}(\delta_i^2 y_j + 2\delta_i y_j l_i); \quad C = \frac{\delta_i}{S_i}$$

$$\delta_i = x_{i+1} - x_i; \quad l_i = x_j - x_i$$

where I_i is the area moment of inertia of the i-th element and S_i is the cross-section area of the teeth at element i, and W is tooth width

Gear meshing stiffness (cont'd)

- The effective modulus of elasticity E_e is defined to take into account of the influence of width of the teeth profile, which is

$$E_e = \frac{E}{1 - v^2}$$

where E is the material's Young modulus and v is the Poisson's ratio.

- The stiffness of the gear meshing is the inverse of combined compliance of two gears at the engaging position

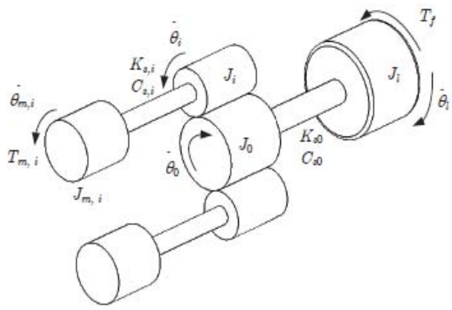
$$\frac{1}{k} = \frac{1}{k_{g,i}} + \frac{1}{k_{g,j}} \qquad (3.5)$$

- Alternatively, the meshing stiffness can be determined through FEA

Case Study: Model with flexible gear teeth and shaft [5]

Assumptions:

- consisting of one pair of spur gears
- include geometric variations like backlash



$$J_{m,i}\ddot{\theta}_{m,i} + C_{s,i}(\dot{\theta}_{m,i} - \dot{\theta}_i) + K_{s,i}(\theta_{m,i} - \theta_i) = T_{m,i}, \quad i = 1, \ldots, l$$

$$J_i\ddot{\theta}_i + C_{s,i}(\dot{\theta}_i - \dot{\theta}_{m,i}) + K_{s,i}(\theta_i - \theta_{m,i}) + F_{b,i}R_{b,i} = 0, \quad i = 1, \ldots, N$$

$$J_0\ddot{\theta}_0 + C_{s0}(\dot{\theta}_0 - \dot{\theta}_l) + K_{s0}(\theta_0 - \theta_l) - \sum_{j=1}^{N} F_{b,j}R_{b0} = 0$$

$$J_l\ddot{\theta}_l + C_{s0}(\dot{\theta}_l - \dot{\theta}_0) + K_{s0}(\theta_l - \theta_0) = -T_f$$

where

J_m and J_l — moments of inertia of the motor and the equivalent load

$R_{b,i}$ and $R_{b,j}$ — radii of the base circles of the driving gear and the driven gears

C_{sm} and K_{sm} — damping coefficient and the stiffness of the motor shaft

C_{sl} and K_{sl} — damping factor and the stiffness of the shaft connected to the load

$F_{b,i}$ and $F_{b,j}$— gear meshing force along the line of action

Case study (cont'd)

- By introducing the dynamic transmission errors, and relative rotations, we can eliminate one degree of freedom

$$u_i = R_{b,i}\theta_i - R_{b0}\theta_0 - e_i, \quad i = 1, \dots, N$$

$$\phi_0 = \theta_0 - \theta_l \text{ and } \phi_i = \theta_{m,i} - \theta_i$$

⇩

$$J_{e,i}\ddot{\phi}_i + C_{s,i}\dot{\phi}_i + K_{s,i}\phi_i + \alpha_i(c_i\dot{u}_i + \bar{k}_i u_i)R_{b,i} = f_{\phi,i}, \quad i = 1, \dots, N$$

$$m_{e,i}\ddot{u}_i + c_i\dot{u}_i + \beta_i \sum c_j\dot{u}_j + (1-\beta_i)C_{s,i}\dot{\phi}_i/R_{b,i} - \beta_i C_{s0}\dot{\phi}_0/R_{b0}$$
$$+\bar{k}u_i + \beta_i \sum \bar{k}u_j + (1-\beta_i)K_{s,i}\phi_i/R_{b,i} - \beta_i K_{s0}\phi_0/R_{b0} = f_{u,i}, \quad i = 1, \dots, N$$

$$J_{e0}\ddot{\phi}_0 + C_{s0}\dot{\phi}_0 + K_{s0}\phi_0 - \alpha_0 \sum (c_j\dot{u}_j + \bar{k}_j u_j)R_{b0} = f_{\phi 0}$$

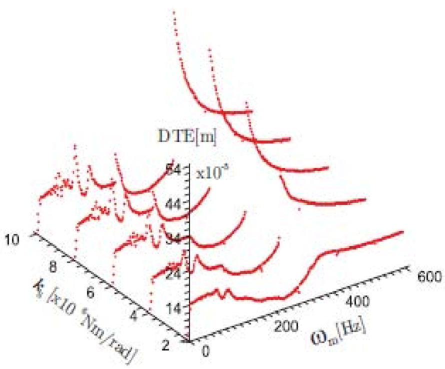
$$\boxed{\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{f}} \qquad \mathbf{q} = \{\phi_1, \dots, \phi_N, u_1, \dots, u_N, \phi_0\} \qquad (3.6)$$

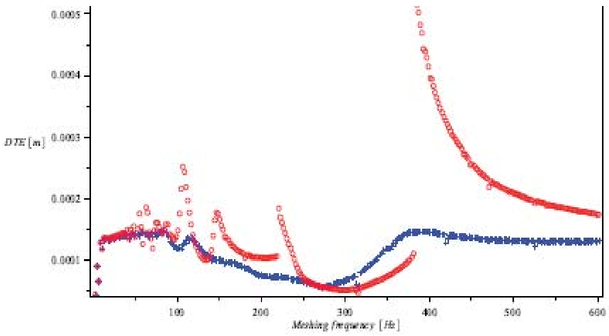
Case Study (cont'd)
Simulaitons: Nonlinearity analysis

- Frequency response

Table 1. Gear Parameters

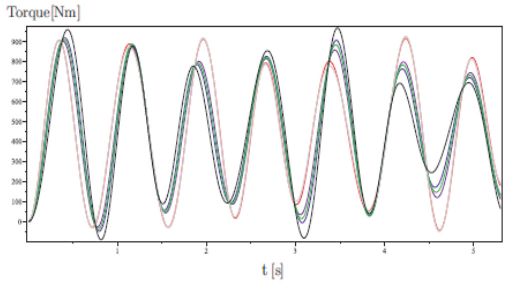
	pinon	gear
Number of teeth	18	35
module[mm]	15	15
pressure angle[deg]	20	20
MOI [kg.m^2]	0.55	7.85
profile error [μm]	$E_t = 209, E_p = 72$	
meshing stiffness [N/m]	$\bar{k}_i = 3.5 \times 10^7; \quad k_r = 2.0 \times 10^7$	
shaft stiffness [Nm/rad]	$K_{s,i} = k_s; \quad K_{s0} = 10 \times k_s$	



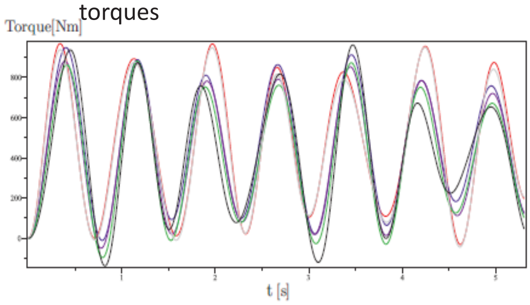


Nonllinearity is weakened with the decreasing of shaft torsional stiffness

Case study (cont'd)
Load-sharing simulation

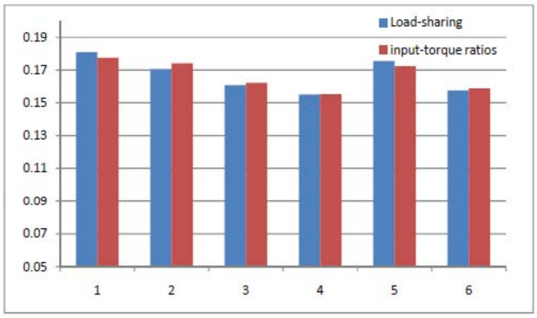


Time history with identical input torques



Time history with slightly different input torques

$$LS_i = RMS(\frac{T_{q,i}(t)}{\sum T_{q,i}(t)}) \quad (3.7)$$



The load-sharing is not sensitive to difference between the input torques

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