LECTURE NOTES

Flexible multibody dynamics

1 INTRODUCTION

Simulation is an abstract theme which can be used to describe an imitative action of a real system. In this paper, simulation is comprised of a computer-aided approach to analyze complex mechanical systems such as mobile and industrial machines. A common feature of these machines is that they include mechanical components as well as various actuators and control schemes. In order to simulate a mechanical system using computers, a mathematical description of the system – a simulation model – needs to be formulated. The simulation model may include sub-models such as hydraulic, pneumatic or electrical drives. These actuators are usually important in terms of the dynamic performance of the machines.

1.1 Overview of multibody system dynamics

A multibody system consists of rigid and flexible bodies, joint constraints that couple the bodies, and power components describing dampers, springs and actuators. Depending on the components needed for the multibody model, the dynamic behavior of the system can be described by a system of equations consisting of differential and nonlinear algebraic equations. In a historical timeline, multibody system analysis has been developed based on the achievements of classical mechanics, which is generally divided into two branches. In the first branch, which can be referred to as the direct approach to dynamics, force and momentum are considered as the primary parameters in differential equations of motion. This form of dynamical equation can be directly derived by employing the approach of Newton and Euler. The second branch is called the indirect or variational approach where forces that perform no work can be neglected. D'Alembert studied a set of rigid bodies introducing the concept of virtual work. In order to make the concept mathematically consistent, Lagrange utilized the results of d'Alembert, making possible the systematic analysis of a constrained particle system. Subsequently, the invention of digital computers made it possible to reformulate these achievements, leading to multibody formalisms in the 1960s [1]. Probably the best-known method in the field of multibody dynamics is the method of Lagrange multipliers, which can be derived from the variational approach. When Newton-Euler equations are used, the linear and angular momentum principles can be utilized directly in formulating equations of motion, whereas the free body principle can be used to solve the reaction forces due to the constraints. However, the use of free body diagrams in large systems is laborious, making the approach vulnerable to human error. Fortunately, the Newton-Euler equations can be derived from the Lagrange equation using the variational approach and the centroidal body reference frame.

Accordingly, constraints can be taken into account by applying the Lagrange multiplier theorem to the variational form of Newton-Euler equations [1].

Flexible multibody dynamics

Multibody dynamics analyses frequently require that structural flexibility is accounted for in order to reliably predict the dynamic behavior of slender structures under a heavy load. It is noteworthy that even though the topological structure of models remains unchanged in the case of rigid and flexible bodies, the modeling of systems with flexible bodies is remarkably challenging regardless of the method used for describing the flexibility [2].

Common techniques to describe the elasticity of the bodies are the lumped mass technique and the floating frame of reference formulation. In the lumped mass technique, the body is divided into rigid segments which are interconnected by force elements. The method is easy to implement in simulation software based on the multibody approach due to the fact that each segment can be treated as a rigid body. However, after segmentation, each flexible body contains several rigid bodies increasing the degrees of the freedom of the system. In practice, the method can be used to describe beam type bodies. In this thesis, structural flexibility is accounted for by using the floating frame of reference formulation. In the method, the generalized coordinates that define the configuration of the flexible body can be divided into ones that describe the position and orientation of the reference coordinate system and ones that describe deformations with respect to the reference coordinate system. In the floating frame of reference formulation, deformations are usually described using methods based on the finite element approach. The first general purpose implementation of the floating frame of reference formulation applicable to large flexible multibody systems in planar cases was introduced by Song and Haug [3]. They used nodal coordinates from finite element discretization to describe deformations. Nevertheless, in that study, the implementation was cumbersome especially for geometrically complex bodies, leading to computationally expensive equations of motion due to a need for a large number of nodal coordinates. To reduce the number of coordinates related to flexibility, Shabana [4] extended the floating frame of reference formulation to three-dimensional mechanisms, and proposed the use of component mode synthesis to extract the structural vibration modes. In this way, the set of nodal coordinates from the finite element method can be replaced by a lower number of modal coordinates, making the numerical solution of the equations of motion more efficient. However, the general purpose application of the approach was impeded because elements used in the modeling of flexible bodies were included in the solution algorithm leading to element-specific volume integrals to be solved. Yoo and Haug [5, 6] introduced the use of static correction modes in order to account for local deformations due to joint constraints and force components. The advantage of the method is that it allows vibration and static correction modes to be solved directly using commercial finite element software.

Constraint modeling

Creating a general-purpose multibody algorithm that takes structural flexibility into account is a challenging endeavor. One of the most difficult tasks in the implementation is to create a component library, which is needed for taking kinematic joint constraints into consideration. References [7, 8] introduce an approach which models joint constraints by using virtual bodies. In this approach, the constraint equations are developed between massless rigid bodies. The advantage of this approach is its applicability to be used in different descriptions of flexibility. On the other hand, adding virtual bodies increases the computation time compared to methods which derive joint constraints individually for each approach to describing flexibility. The formulation of kinematic joints composed of simple basic constraints in the case of systems of rigid bodies has been discussed in References [9, 10]. The basic constraint equations for modeling spherical, universal and revolute joints between flexible bodies have been presented in Reference [5]. Shabana [11, 12] has introduced an approach based on intermediate body fixed joint coordinate systems which are rigidly attached to joint definition points. In this approach, the joint coordinate systems are used to derive basic constraint equations including sliding joints with the assumption that the joint axis can be described as a rigid line. Cardona [13] has introduced the finite element approach for mechanical joints, which can be integrated into finite element software. In Reference [14], the basic joint constraints were used in the context of topological multibody formulation. Hwang [15] has presented basic constraint types used with translational joint models which account for the deformation of the axis line. Hwang used the floating frame of reference approach accounting for multiple contact points, whereas the numerical results are only shown in the case of a single contact point.

In order to be able to employ traditional solvers for the Ordinary Differential Equation (ODE) within the system of equations, the constraint equations must be differentiated twice with respect to time. It is important to note that in previous literature, the terms of the Jacobian matrix and terms that are related to second time differentials of basic constraint equations are not explicitly presented.

In order to alleviate the development of modular simulation, the components that are required to take constraints into account need to be obtained.

2 MODELING OF MULTIBODY SYSTEMS USING THE REFERENCE FRAME APPROACH

The method of the floating frame of reference is the method most frequently applied to describe linear deformations in multibody applications. This is due to the computational efficiency of the method and the possibility to utilize commercial finite element software to define properties of flexible bodies. In this chapter, the floating frame of reference approach with three different descriptions of equations of motion is briefly introduced.

2.1 Spatial kinematics of a flexible body

The floating frame of reference formulation can be applied to bodies that experience large rigid body translations and rotations as well as elastic deformations. The method is based on describing deformations of a flexible body with respect to a frame of reference. The frame of reference, in turn, is employed to describe large translations and rotations. The deformations of a flexible body with respect to its frame of reference can be described with a number of methods, whereas in this study, deformation is described using linear deformation modes of the body. Deformation modes can be defined using a finite element model of the body. Fig. 1 illustrates the position of particle P^i in a deformed body *i*.

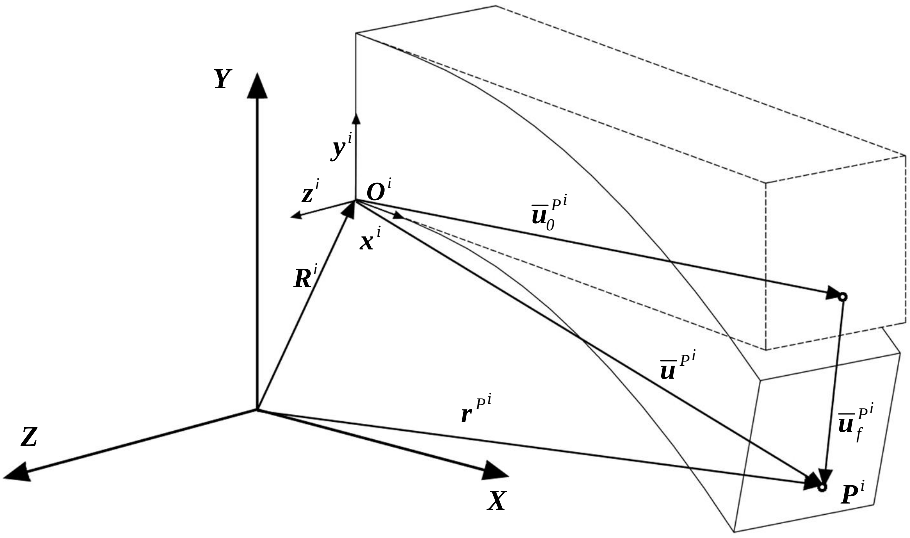


Figure 1. The position of the particle P^i in global coordinate system.

The position of particle P^i of the flexible body *i* can be described in a global coordinate system using the vector $\boldsymbol{r}^{P^i}$ as follows:

$$\boldsymbol{r}^{P^i} = \boldsymbol{R}^i + \mathbf{A}^i \bar{\boldsymbol{u}}^{P^i} = \boldsymbol{R}^i + \mathbf{A}^i \left(\bar{\boldsymbol{u}}_0^{P^i} + \bar{\boldsymbol{u}}_f^{P^i} \right), \tag{1}$$

where $\boldsymbol{R}^i$ is the position vector of the frame of reference, $\mathbf{A}^i$ is the rotation matrix of body *i*, $\bar{\boldsymbol{u}}^{P^i}$ is the position vector of particle P^i within the frame of reference, $\bar{\boldsymbol{u}}_0^{P^i}$ is the undeformed position vector of the particle within the frame of reference, and $\bar{\boldsymbol{u}}_f^{P^i}$ is the displacement of particle P^i within the frame of reference due to the deformation of body *i*. In this study, the rotation matrix $\mathbf{A}^i$ is expressed using Euler parameters $\boldsymbol{\theta}^{E^i \mathrm{T}} = \left[\theta_0^{E^i} \quad \theta_1^{E^i} \quad \theta_2^{E^i} \quad \theta_3^{E^i} \right]^{\mathrm{T}}$ in order to avoid singular conditions which are a problem when three rotational parameters are used, such as in the cases of Euler and Bryant angles [16]. The rotation matrix can be written using Euler parameters as follows:

$$\mathbf{A}^i = 2 \begin{bmatrix} \frac{1}{2} - \left(\theta_2^{E^i}\right)^2 - \left(\theta_3^{E^i}\right)^2 & \theta_1^{E^i}\theta_2^{E^i} - \theta_0^{E^i}\theta_3^{E^i} & \theta_1^{E^i}\theta_3^{E^i} + \theta_0^{E^i}\theta_2^{E^i} \\ \theta_1^{E^i}\theta_2^{E^i} + \theta_0^{E^i}\theta_3^{E^i} & \frac{1}{2} - \left(\theta_1^{E^i}\right)^2 - \left(\theta_3^{E^i}\right)^2 & \theta_2^{E^i}\theta_3^{E^i} - \theta_0^{E^i}\theta_1^{E^i} \\ \theta_1^{E^i}\theta_3^{E^i} - \theta_0^{E^i}\theta_2^{E^i} & \theta_2^{E^i}\theta_3^{E^i} + \theta_0^{E^i}\theta_1^{E^i} & \frac{1}{2} - \left(\theta_1^{E^i}\right)^2 - \left(\theta_2^{E^i}\right)^2 \end{bmatrix}. \tag{2}$$

The following mathematical constraint must be taken into consideration when Euler parameters are applied:

$$\left(\theta_0^{E^i}\right)^2+\left(\theta_1^{E^i}\right)^2+\left(\theta_2^{E^i}\right)^2+\left(\theta_3^{E^i}\right)^2=1 . \tag{3}$$

The deformation vector $\bar{\boldsymbol{u}}_f^{P^i}$ can be described using a linear combination of the deformation modes as follows:

$$\bar{\boldsymbol{u}}_f^{P^i}=\boldsymbol{\psi}_R^{P^i}\boldsymbol{q}_f^i , \tag{4}$$

where $\boldsymbol{\psi}_R^{P^i}$ is the modal matrix whose columns describe the translation of particle P^i within the assumed deformation modes of the flexible body i [11], and $\boldsymbol{q}_f^i$ is the vector of elastic coordinates. Consequently, the generalized coordinates that uniquely define the position of point P^i can be represented with vector $\boldsymbol{p}^i$ as follows:

$$\boldsymbol{p}^{i\mathrm{T}}=\begin{bmatrix}\boldsymbol{R}^{i\mathrm{T}} & \boldsymbol{\theta}^{E^i\mathrm{T}} & \boldsymbol{q}_f^{i\,\mathrm{T}}\end{bmatrix}^{\mathrm{T}} . \tag{5}$$

The velocity of particle P^i can be obtained by differentiating the position description (Eq. 1) with respect to time as follows:

$$\dot{\boldsymbol{r}}^{P^i}=\dot{\boldsymbol{R}}^i-\mathbf{A}^i\left(\tilde{\bar{\boldsymbol{u}}}_0^{P^i}+\tilde{\boldsymbol{\psi}}_R^{P^i}\boldsymbol{q}_f^i\right)\bar{\boldsymbol{\omega}}^i+\mathbf{A}^i\boldsymbol{\psi}_R^{P^i}\dot{\boldsymbol{q}}_f^i , \tag{6}$$

where $\bar{\boldsymbol{\omega}}^i$ is the vector of local angular velocities. In Eq. 6, the generalized velocity vector can be defined as follows:

$$\dot{\boldsymbol{q}}^{i\mathrm{T}}=\begin{bmatrix}\dot{\boldsymbol{R}}^{i\mathrm{T}} & \bar{\boldsymbol{\omega}}^{i\mathrm{T}} & \dot{\boldsymbol{q}}_f^{i\,\mathrm{T}}\end{bmatrix}^{\mathrm{T}} . \tag{7}$$

By differentiating Eq. (6) with respect to time, the following formulation for the acceleration of particle P^i can be obtained:

$$\ddot{\boldsymbol{r}}^{P^i}=\ddot{\boldsymbol{R}}^i+\mathbf{A}^i\tilde{\bar{\boldsymbol{\omega}}}^i\tilde{\bar{\boldsymbol{\omega}}}^i\bar{\boldsymbol{u}}^{P^i}+\mathbf{A}^i\dot{\tilde{\bar{\boldsymbol{\omega}}}}^i\bar{\boldsymbol{u}}^{P^i}+2\mathbf{A}^i\tilde{\bar{\boldsymbol{\omega}}}^i\dot{\bar{\boldsymbol{u}}}^{P^i}+\mathbf{A}^i\ddot{\bar{\boldsymbol{u}}}^{P^i} , \tag{8}$$

where $\tilde{\bar{\boldsymbol{\omega}}}^i$ is a skew-symmetric representation of the angular velocity of the body in the frame of reference, $\ddot{\boldsymbol{R}}^i$ is the vector that defines the translational acceleration of the frame of reference, $\mathbf{A}^i\tilde{\bar{\boldsymbol{\omega}}}^i\tilde{\bar{\boldsymbol{\omega}}}^i\bar{\boldsymbol{u}}^{P^i}$ is the normal component of acceleration, $\mathbf{A}^i\dot{\tilde{\bar{\boldsymbol{\omega}}}}^i\bar{\boldsymbol{u}}^{P^i}$ is the tangential component of acceleration, $2\mathbf{A}^i\tilde{\bar{\boldsymbol{\omega}}}^i\dot{\bar{\boldsymbol{u}}}^{P^i}$ is the Coriolis component of acceleration and $\mathbf{A}^i\ddot{\bar{\boldsymbol{u}}}^{P^i}$ is the acceleration of particle P^i due to the deformation of body i.

When deformation modes are used with the floating frame of reference, rotations due to body deformation are usually ignored. However, in order to compose all of the basic constraints, rotation due to body deformation must be accounted for. The vector $\bar{\boldsymbol{v}}_f^i$ due to deformation at the location of particle P^i within the frame of reference can be expressed as follows:

$$\bar{\boldsymbol{v}}_f^i = \mathbf{A}_f^{P^i} \bar{\boldsymbol{v}}^i , \tag{9}$$

where $\bar{\boldsymbol{v}}^i$ is defined in the undeformed state at the location of particle P^i, and $\mathbf{A}_f^{P^i}$ is a rotation matrix that describes the orientation due to deformation at the location of particle P^i with respect to the reference frame. Note that all components in Eq. 9 are expressed in the reference frame. The rotation matrix $\mathbf{A}_f^{P^i}$ can be expressed as follows:

$$\mathbf{A}_f^{P^i} = \mathbf{I} + \tilde{\boldsymbol{\varepsilon}}^{P^i} , \tag{10}$$

where $\mathbf{I}$ is a (3×3) identity matrix and $\tilde{\boldsymbol{\varepsilon}}^{P^i}$ is a skew symmetric form of the rotation change caused by deformation. Rotation changes due to deformation can be represented in the following way:

$$\boldsymbol{\varepsilon}^{P^i} = \boldsymbol{\psi}_\theta^{P^i} \boldsymbol{q}_f^i , \tag{11}$$

where $\boldsymbol{\psi}_\theta^{P^i}$ is the modal transformation matrix whose columns describe rotation coordinates of point P^i within the assumed deformation modes of the flexible body *i* [11], and $\boldsymbol{q}_f^i$ is the vector of elastic coordinates.

2.2 Virtual work

The equations of motion can be developed using the principle of virtual work, which can be written for inertia forces as follows:

$$\delta W^{i^i} = \int_{V^i} \rho^i \delta \boldsymbol{r}^{P^i \mathrm{T}} \ddot{\boldsymbol{r}}^{P^i} \, dV^i , \tag{12}$$

where $\delta \boldsymbol{r}^{P^i}$ is the virtual displacement of the position vector of a particle, $\ddot{\boldsymbol{r}}^{P^i}$ is the acceleration vector of the particle defined in Eq 8, ρ^i is density of body *i*, and V^i is volume of body *i*. Accordingly, the virtual displacement of the position vector can be expressed in terms of virtual displacement of generalized coordinates as follows:

$$\delta \boldsymbol{r}^{P^i \mathrm{T}} = \begin{bmatrix} \delta \boldsymbol{R}^{i\mathrm{T}} & \delta \boldsymbol{\theta}^{i\mathrm{T}} & \delta \boldsymbol{q}_f^{i\,\mathrm{T}} \end{bmatrix} \begin{bmatrix} \mathbf{I} \\ -\tilde{\bar{\boldsymbol{u}}}^{P^i \mathrm{T}} \mathbf{A}^{i\mathrm{T}} \\ \boldsymbol{\psi}_R^{P^i \mathrm{T}} \mathbf{A}^{i\mathrm{T}} \end{bmatrix} , \tag{13}$$

where $\delta\boldsymbol{\theta}^i$ is virtual rotation. By substituting the virtual displacement of the position vector (13) into the equation of virtual work of the inertial forces (12) and by separating the terms related to acceleration from the terms related quadratically to velocities, the following equation for the virtual work of inertial forces can be obtained:

$$\delta W^{i^i} = \delta \boldsymbol{q}^i \left[\mathbf{M}^i \ddot{\boldsymbol{q}}^i + \boldsymbol{Q}^{v^i} \right] \ , \tag{14}$$

where $\mathbf{M}^i$ is the mass matrix and $\boldsymbol{Q}^{v^i}$ is the quadratic velocity vector. The mass matrix can be expressed as follows:

$$\mathbf{M}^i = \int_{V^i} \rho^i \begin{bmatrix} \mathbf{I} & -\mathbf{A}^i \tilde{\bar{\boldsymbol{u}}}^{P^i} & \mathbf{A}^i \boldsymbol{\psi}_R^{P^i} \\ & \tilde{\bar{\boldsymbol{u}}}^{P^i\mathrm{T}} \tilde{\bar{\boldsymbol{u}}}^{P^i} & -\tilde{\bar{\boldsymbol{u}}}^{P^i\mathrm{T}} \boldsymbol{\psi}_R^{P^i} \\ sym & & \boldsymbol{\psi}_R^{P^i\mathrm{T}} \boldsymbol{\psi}_R^{P^i} \end{bmatrix} dV^i \ . \tag{15}$$

And, correspondingly, the quadratic velocity vector takes the form

$$\boldsymbol{Q}^{v^i} = \int_{V^i} \rho^i \begin{bmatrix} \mathbf{A}^i \tilde{\bar{\boldsymbol{\omega}}}^i \tilde{\bar{\boldsymbol{\omega}}}^i \bar{\boldsymbol{u}}^{P^i} + 2\mathbf{A}^i \tilde{\bar{\boldsymbol{\omega}}}^i \boldsymbol{\psi}_R^{P^i} \dot{\boldsymbol{q}}_f^i \\ -\tilde{\bar{\boldsymbol{u}}}^{P^i\mathrm{T}} \tilde{\bar{\boldsymbol{\omega}}}^i \tilde{\bar{\boldsymbol{\omega}}}^i \bar{\boldsymbol{u}}^{P^i} - 2\tilde{\bar{\boldsymbol{u}}}^{P^i\mathrm{T}} \tilde{\bar{\boldsymbol{\omega}}}^i \boldsymbol{\psi}_R^{P^i} \dot{\boldsymbol{q}}_f^i \\ \boldsymbol{\psi}_R^{P^i\mathrm{T}} \tilde{\bar{\boldsymbol{\omega}}}^i \tilde{\bar{\boldsymbol{\omega}}}^i \bar{\boldsymbol{u}}^{P^i} + 2\boldsymbol{\psi}_R^{P^i\mathrm{T}} \tilde{\bar{\boldsymbol{\omega}}}^i \boldsymbol{\psi}_R^{P^i} \dot{\boldsymbol{q}}_f^i \end{bmatrix} dV^i \tag{16}$$

The virtual work of the externally applied forces can be written as:

$$\delta W^{e^i} = \int_{V^i} \delta \boldsymbol{r}^{P^i\mathrm{T}} \boldsymbol{F}^{P^i} dV^i = \delta \boldsymbol{q}^{i\mathrm{T}} \boldsymbol{Q}^{e^i}, \tag{17}$$

where $\boldsymbol{F}^{P^i}$ is external force per unit mass and $\boldsymbol{Q}^{e^i}$ is the vector of generalized forces which can be expressed as follows:

$$\boldsymbol{Q}^{e^i} = \begin{bmatrix} \sum_{j=1}^{n_F} \boldsymbol{F}_j^i \\ \sum_{j=1}^{n_F} \tilde{\bar{\boldsymbol{u}}}_j^i \mathbf{A}^{i\mathrm{T}} \boldsymbol{F}_j^i \\ \sum_{j=1}^{n_F} \boldsymbol{\psi}_j^{i\,\mathrm{T}} \mathbf{A}^{i\mathrm{T}} \boldsymbol{F}_j^i \end{bmatrix} \tag{18}$$

where $\boldsymbol{F}_j^i$ is the *j*-th force component acting on body *i*, $\tilde{\bar{\boldsymbol{u}}}_j^i$ is a skew symmetric matrix of the location vector of the *j*-th force components, and $\boldsymbol{\psi}_j^i$ is the terms of the modal matrix associated with the node to which the *j*-th force component applies.

The elastic forces can be defined using the modal stiffness matrix $\mathbf{K}^i$ and modal coordinates. The modal stiffness matrix is associated with the modal coordinates and the matrix can be obtained from the conventional finite element approach using the component mode synthesis technique [11]. The virtual work of elastic forces can be written as follows:

$$\delta W^{s^i} = \delta \boldsymbol{q}_f^{i\,\mathrm{T}}\, \mathbf{K}^i\, \boldsymbol{q}_f^i \,. \tag{19}$$

Accordingly, the vector of elastic forces can be represented as follows:

$$\boldsymbol{Q}^{f^i} = \begin{bmatrix} 0 \\ 0 \\ \mathbf{K}^i\, \boldsymbol{q}_f^i \end{bmatrix}. \tag{20}$$

Using Eqs. 14, 17 and 19, the equation of virtual work including inertial, external and internal force components can be written as follows:

$$\delta \boldsymbol{q}^i \left[\mathbf{M}^i\, \ddot{\boldsymbol{q}}^i + \boldsymbol{Q}^{v^i} + \boldsymbol{Q}^{f^i} - \boldsymbol{Q}^{e^i}\right] = 0 \,. \tag{21}$$

The terms inside the brackets can be used to form unconstrained Newton-Euler equations as follows:

$$\begin{bmatrix} \int_{V^i} \rho^i \mathbf{I} dV^i & -\int_{V^i} \rho^i \mathbf{A}^i \tilde{\bar{\boldsymbol{u}}}^{P^i} dV^i & \int_{V^i} \rho^i \mathbf{A}^i \boldsymbol{\psi}_R^{P^i} dV^i \\ & \int_{V^i} \rho^i\, \tilde{\bar{\boldsymbol{u}}}^{P^i\mathrm{T}} \tilde{\bar{\boldsymbol{u}}}^{P^i} dV^i & -\int_{V^i} \rho^i\, \tilde{\bar{\boldsymbol{u}}}^{P^i\mathrm{T}}\, \boldsymbol{\psi}_R^{P^i} dV^i \\ sym & & \int_{V^i} \rho^i \boldsymbol{\psi}_R^{P^i\mathrm{T}}\, \boldsymbol{\psi}_R^{P^i} dV^i \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{R}}^i \\ \dot{\bar{\boldsymbol{\omega}}}^i \\ \ddot{\boldsymbol{q}}_f^i \end{bmatrix}$$

$$= \begin{bmatrix} \int_{V^i} \boldsymbol{F}^{P^i} dV^i \\ \int_{V^i} \tilde{\bar{\boldsymbol{u}}}^{P^i}\, \mathbf{A}^{i\mathrm{T}}\, \boldsymbol{F}^{P^i} dV^i \\ \int_{V^i} \boldsymbol{\psi}_R^{P^i\mathrm{T}}\, \mathbf{A}^{i\mathrm{T}}\, \boldsymbol{F}^{P^i} dV^i \end{bmatrix} - \begin{bmatrix} \int_{V^i} \rho^i \left(\mathbf{A}^i\, \tilde{\bar{\boldsymbol{\omega}}}^i \tilde{\bar{\boldsymbol{\omega}}}^i \bar{\boldsymbol{u}}^{P^i} + 2\mathbf{A}^i\, \tilde{\bar{\boldsymbol{\omega}}}^i\, \boldsymbol{\psi}_R^{P^i}\, \dot{\boldsymbol{q}}_f^i\right) dV^i \\ \int_{V^i} \rho^i \left(-\tilde{\bar{\boldsymbol{u}}}^{P^i\mathrm{T}}\, \tilde{\bar{\boldsymbol{\omega}}}^i \tilde{\bar{\boldsymbol{\omega}}}^i \bar{\boldsymbol{u}}^{P^i} - 2\tilde{\bar{\boldsymbol{u}}}^{P^i\mathrm{T}}\, \tilde{\bar{\boldsymbol{\omega}}}^i\, \boldsymbol{\psi}_R^{P^i}\, \dot{\boldsymbol{q}}_f^i\right) dV^i \\ \int_{V^i} \rho^i \left(\boldsymbol{\psi}_R^{P^i\mathrm{T}}\, \tilde{\bar{\boldsymbol{\omega}}}^i \tilde{\bar{\boldsymbol{\omega}}}^i \bar{\boldsymbol{u}}^{P^i} + 2\boldsymbol{\psi}_R^{P^i\mathrm{T}}\, \tilde{\bar{\boldsymbol{\omega}}}^i\, \boldsymbol{\psi}_R^{P^i}\, \dot{\boldsymbol{q}}_f^i\right) dV^i \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ \mathbf{K}^i\, \boldsymbol{q}_f^i \end{bmatrix} \tag{22}$$

Equations of motion in this form are referred to as Generalized Newton-Euler equations in Reference [11], where Newton-Euler equations of rigid bodies are extended to flexible bodies.

2.2.1 Integration of the equations of motion

Due to the use of Generalized Newton-Euler equations as a description of dynamics, the equations of motion are expressed using the angular velocity and angular acceleration vectors. Eq. 22 can be solved for angular accelerations in the body frame which can be integrated with angular velocities.

However, the problem arises when the coordinates describing the orientation of the body have to be solved. This is due to the fact that angular velocities cannot be directly integrated with the parameters which uniquely describe the orientation of the body. For this reason, a new set of variables $\boldsymbol{p}$ is defined, containing the orientation coordinates of the body reference frame. In order to integrate the position level coordinates, the first time derivative of Euler parameters and the vector of angular velocities defined in the body reference frame can be related through the following linear expression:

$$\dot{\boldsymbol{\theta}}^{E^i} = \tfrac{1}{2}\overline{\mathbf{G}}^{i\mathrm{T}}\overline{\boldsymbol{\omega}}^i, \tag{23}$$

where the velocity transformation matrix $\overline{\mathbf{G}}^i$ can be written as follows:

$$\overline{\mathbf{G}}^i = \begin{bmatrix} -\theta_1^{E^i} & \theta_0^{E^i} & \theta_3^{E^i} & -\theta_2^{E^i} \\ -\theta_2^{E^i} & -\theta_3^{E^i} & \theta_0^{i} & \theta_1^{E^i} \\ -\theta_3^{E^i} & \theta_2^{E^i} & -\theta_1^{E^i} & \theta_0^{E^i} \end{bmatrix}. \tag{24}$$

The time derivatives of the body variables to be intergrated can be stated using vector $\dot{\boldsymbol{p}}$ as follows:

$$\dot{\boldsymbol{p}}^{i\mathrm{T}} = \begin{bmatrix} \dot{\boldsymbol{R}}^{i\mathrm{T}} & \dot{\boldsymbol{\theta}}^{E^i\mathrm{T}} & \dot{\boldsymbol{q}}_f^{i\,\mathrm{T}} \end{bmatrix}^{\mathrm{T}}, \tag{25}$$

which can be integrated to obtain position level generalized coordinates $\boldsymbol{p}$.

2.3 Description of multibody equations of motion

In this section, the three multibody formalisms used in this work are briefly described. The formalisms discussed here are the method based on Lagrange multipliers, which is also referred to as the descriptor form [17, 11], the penalty and augmented Lagrangian methods [18, 19] and the method based on projection matrix [20, 21, 22].

2.3.1 Method of Lagrange multipliers

When constraint equations are augmented to equations of motion using the Lagrange multiplier technique, the result can be written as:

$$\mathbf{M}\ddot{\boldsymbol{q}} + \mathbf{C}_{\boldsymbol{q}}^{\mathrm{T}}\lambda = \boldsymbol{Q}^e - \boldsymbol{Q}^v - \boldsymbol{Q}^f, \tag{26}$$

where $\boldsymbol{q}$ is the vector of n generalized coordinates that define the position and orientation of each body in the system, $\mathbf{M}$ is the mass matrix, $\boldsymbol{Q}^e$ is the vector of generalized forces, $\boldsymbol{Q}^v$ is the quadratic velocity vector that includes velocity dependent inertia forces, $\mathbf{C}_{\boldsymbol{q}}$ is the Jacobian matrix of the constraint equations, $\boldsymbol{Q}^f$ is the vector of elastic forces and $\boldsymbol{\lambda}$ is the vector of Lagrange multipliers. To satisfy a set of m constraint equations related to generalized coordinates, the following equation must be fulfilled:

$$\boldsymbol{C}(\boldsymbol{q}, t) = \mathbf{0}, \tag{27}$$

where $\boldsymbol{C}$ is a vector of constraints of the system and t is time. Eqs. (26) and (27) comprise a system of differential algebraic equations (DAE) which describe the dynamical behavior of the mechanics. In order to solve the set of equations using ordinary integration methods for differential equation (ODE), the equations must be transformed to the second order ODE. For this reason, Eq. (27) is differentiated twice with respect to time:

$$\ddot{\boldsymbol{C}}(\boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}, t) = \mathbf{C}_{\boldsymbol{q}}\, \ddot{\boldsymbol{q}} + \boldsymbol{Q}^c = \mathbf{0}, \tag{28}$$

where $\boldsymbol{Q}^c$ includes velocity dependent terms due to differentation. By combining Eqs. (26) and (28), the matrix representation of equations of motion can be obtained as follows:

$$\begin{bmatrix} \mathbf{M} & \mathbf{C}_{\boldsymbol{q}}^{\mathrm{T}} \\ \mathbf{C}_{\boldsymbol{q}} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \ddot{\boldsymbol{q}} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \boldsymbol{Q}^e - \boldsymbol{Q}^v - \boldsymbol{Q}^f \\ -\boldsymbol{Q}^c \end{bmatrix}, \tag{29}$$

where the invertable matrix is of the size $(m+n)\times(m+n)$. The equation of motion can be integrated using the standard ODE solver [11]. However, equations of motion cannot guarantee that constraint equations in Eq. (27) are satisfied. This is due to the fact that during the differentiation of a constraint equation, constant terms disappear, and consequently, Eq. (29) fulfils the constraints at acceleration level only. Therefore, numerical integration causes errors to accumulate in the kinematic joint constraints. To overcome this problem, a stabilization method must be used. Another possibility to solve this problem is to use methods which produce a general solution to differential algebraic equations [19, 23].

2.3.2 Augmented Lagrangian method

In the penalty method, Lagrange multipliers are eliminated from the equations of motion by employing penalty terms. This procedure leads to a set of n differential equations as follows:

$$\left(\mathbf{M} + \mathbf{C}_{\boldsymbol{q}}^{\mathrm{T}} \boldsymbol{\alpha} \mathbf{C}_{\boldsymbol{q}}\right)\ddot{\boldsymbol{q}} = \boldsymbol{Q}^e - \boldsymbol{Q}^v - \boldsymbol{Q}^f - \mathbf{C}_{\boldsymbol{q}}^{\mathrm{T}} \boldsymbol{\alpha}\left(\boldsymbol{Q}^c + 2\boldsymbol{\Omega}\boldsymbol{\mu}\dot{\boldsymbol{C}} + \boldsymbol{\Omega}^2 \boldsymbol{C}\right), \tag{30}$$

where $\boldsymbol{\alpha}$, $\boldsymbol{\Omega}$ and $\boldsymbol{\mu}$ are $m \times m$ diagonal matrices which contain penalty terms, natural frequencies and damping ratios for constraints, respectively. If the penalty terms are equivalent to each constraint, the matrices are identity matrices multiplied with a constant penalty factor.

A drawback associated with the penalty method is that large numerical values for penalty factors must be used, which may lead to numerical ill-conditioning and round-off errors. However, the method can be improved by adding penalty terms or correction terms which are zero when constraint equations are fulfilled. Using this approach, equations of motion can be written as follows:

$$\left(\mathbf{M}+\mathbf{C}_{\boldsymbol{q}}^{\mathrm{T}}\boldsymbol{\alpha}\mathbf{C}_{\boldsymbol{q}}\right)\ddot{\boldsymbol{q}}=\boldsymbol{Q}^{e}-\boldsymbol{Q}^{v}-\boldsymbol{Q}^{f}-\mathbf{C}_{\boldsymbol{q}}^{\mathrm{T}}\boldsymbol{\alpha}\left(\boldsymbol{Q}^{c}+2\boldsymbol{\Omega}\boldsymbol{\mu}\dot{\boldsymbol{C}}+\boldsymbol{\Omega}^{2}\boldsymbol{C}\right)+\mathbf{C}_{\boldsymbol{q}}^{\mathrm{T}}\boldsymbol{\lambda}^{*}, \tag{31}$$

where $\boldsymbol{\lambda}^{*}$ is the vector of penalty forces. By comparing Eqs. (1) and (31), it can be concluded that

$$\boldsymbol{\lambda}=\boldsymbol{\lambda}^{*}-\boldsymbol{\alpha}\left(\mathbf{C}_{\boldsymbol{q}}\ddot{\boldsymbol{q}}+\boldsymbol{Q}^{c}+2\boldsymbol{\Omega}\boldsymbol{\mu}\dot{\boldsymbol{C}}+\boldsymbol{\Omega}^{2}\boldsymbol{C}\right). \tag{32}$$

Since the exact values of $\boldsymbol{\lambda}^{*}$ are not known in advance, an iterative procedure should be used as follows:

$$\boldsymbol{\lambda}_{i+1}^{*}=\boldsymbol{\lambda}_{i}^{*}-\boldsymbol{\alpha}\left(\mathbf{C}_{\boldsymbol{q}}\ddot{\boldsymbol{q}}_{i}+\boldsymbol{Q}^{c}+2\boldsymbol{\Omega}\boldsymbol{\mu}\dot{\boldsymbol{C}}+\boldsymbol{\Omega}^{2}\boldsymbol{C}\right), \tag{33}$$

where $\boldsymbol{\lambda}_{0}^{*}=0$ is used for the first iteration. Using this equation, the forces caused by errors in constraint equations at iteration *i*+1 can be defined and compensated. In this case, the penalty terms do not need to have large numerical values. An iterative procedure can be applied directly to Eq (31), which leads to the following expression:

$$\left(\mathbf{M}+\mathbf{C}_{\boldsymbol{q}}^{\mathrm{T}}\boldsymbol{\alpha}\mathbf{C}_{\boldsymbol{q}}\right)\ddot{\boldsymbol{q}}_{i+1}=\mathbf{M}\ddot{\boldsymbol{q}}_{i}-\mathbf{C}_{\boldsymbol{q}}^{\mathrm{T}}\boldsymbol{\alpha}\left(\boldsymbol{Q}^{c}+2\boldsymbol{\Omega}\boldsymbol{\mu}\dot{\boldsymbol{C}}+\boldsymbol{\Omega}^{2}\boldsymbol{C}\right). \tag{34}$$

In the case of the first iteration, $\mathbf{M}\ddot{\boldsymbol{q}}_{0}=\boldsymbol{Q}^{e}-\boldsymbol{Q}^{v}-\boldsymbol{Q}^{f}$. The leading matrix of Eq. (34) is a symmetric and positive definite, which makes the solution of the equation efficient. This formulation behaves satisfactorily also in singular configurations of a mechanical system.

2.3.3 Method based on projection matrix

The two previously introduced formulations define the equations of motion using a complete set of generalized coordinates. However, the number of the equations can be reduced to the minimum number of differential equations using a set of independent generalized coordinates. Independent generalized velocities $\dot{\boldsymbol{q}}_{i}$ can be defined as a projection of velocities of generalized coordinates $\dot{\boldsymbol{q}}$ using matrix **B** as follows:

$$\dot{\boldsymbol{q}}_{i}=\mathbf{B}\dot{\boldsymbol{q}}. \tag{35}$$

It is noteworthy that the rows of matrix **B** are linearly independent. For skleronomous systems, a solution to describe the transformation from independent generalized coordinates to a complete set of generalized coordinates is available and can be defined using transformation matrix **R** as follows:

$$\dot{\boldsymbol{q}} = \mathbf{R}\dot{\boldsymbol{q}}_i \, . \tag{36}$$

Using coordinate partitioning to dependent $\boldsymbol{q}_d$ and independent $\boldsymbol{q}_i$ generalized coordinates, vector $\boldsymbol{q}$ can be written in the partitioned form $\boldsymbol{q} = \begin{bmatrix}\boldsymbol{q}_d^{\mathrm{T}} & \boldsymbol{q}_i^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}}$. The virtual change of generalized coordinates with respect to constraint equations can be expressed as follows:

$$\mathbf{C}_{\boldsymbol{q}_d}\delta\boldsymbol{q}_d + \mathbf{C}_{\boldsymbol{q}_i}\delta\boldsymbol{q}_i = \mathbf{0}\, , \tag{37}$$

where $\mathbf{C}_{\boldsymbol{q}_d}$ and $\mathbf{C}_{\boldsymbol{q}_i}$ are partitioned Jacobian matrices. $\mathbf{C}_{\boldsymbol{q}_d}$ is a *m* x *m* matrix where *m* is the number of constraint equations. Using Eq. (37), the virtual change of dependent generalized coordinates can be defined as:

$$\delta\boldsymbol{q}_d = -\mathbf{C}_{\boldsymbol{q}_d}^{-1}\mathbf{C}_{\boldsymbol{q}_i}\delta\boldsymbol{q}_i \, . \tag{38}$$

The virtual change of generalized coordinates can now be expressed using independent generalized coordinates as follows:

$$\delta\boldsymbol{q} = \begin{bmatrix}\delta\boldsymbol{q}_d \\ \delta\boldsymbol{q}_i\end{bmatrix} = \begin{bmatrix}-\mathbf{C}_{\boldsymbol{q}_d}^{-1}\mathbf{C}_{\boldsymbol{q}_i} \\ \mathbf{I}\end{bmatrix}\delta\boldsymbol{q}_i \, . \tag{39}$$

Correspondingly, the transformation matrix **R** can be expressed as follows:

$$\mathbf{R} = \begin{bmatrix}-\mathbf{C}_{\boldsymbol{q}_d}^{-1}\mathbf{C}_{\boldsymbol{q}_i} \\ \mathbf{I}\end{bmatrix}. \tag{40}$$

Using coordinate partitioning, accelerations of generalized coordinates can be written as follows:

$$\ddot{\boldsymbol{q}} = \begin{bmatrix}\ddot{\boldsymbol{q}}_d \\ \ddot{\boldsymbol{q}}_i\end{bmatrix} = \begin{bmatrix}-\mathbf{C}_{\boldsymbol{q}_d}^{-1}\mathbf{C}_{\boldsymbol{q}_i} \\ \mathbf{I}\end{bmatrix}\ddot{\boldsymbol{q}}_{\mathrm{i}} + \boldsymbol{\gamma} \tag{41}$$

with the definition

$$\boldsymbol{\gamma} = \begin{bmatrix}-\mathbf{C}_{\boldsymbol{q}_d}^{-1}\left[\left(\mathbf{C}_{\boldsymbol{q}}\dot{\boldsymbol{q}}\right)_{\boldsymbol{q}}\dot{\boldsymbol{q}} + 2\mathbf{C}_{\boldsymbol{q}t}\dot{\boldsymbol{q}} + \boldsymbol{C}_{tt}\right] \\ 0\end{bmatrix}. \tag{42}$$

It can be seen that vector $\boldsymbol{\gamma}$ consist of the accelerations of generalized coordinates when the accelerations of independent coordinates are equal to zero. Using Eq. (42), Eq. (41) can be written as follows:

$$\ddot{\boldsymbol{q}} = \mathbf{R}\ddot{\boldsymbol{q}}_i + \boldsymbol{\gamma} \, . \tag{43}$$

Substitution of the result into Eq. (26) leads to:

$$\mathbf{R}^{\mathrm{T}}\mathbf{M}\mathbf{R}\ddot{\boldsymbol{q}}_i+\mathbf{R}^{\mathrm{T}}\mathbf{M}\boldsymbol{\gamma}-\mathbf{R}^{\mathrm{T}}\left(\boldsymbol{Q}^e-\boldsymbol{Q}^v-\boldsymbol{Q}^f\right)=\mathbf{0}\,. \tag{44}$$

This equation of motion can be solved for independent accelerations which can be integrated to solve the new independent velocities and positions for the next time step. This form of equation of motion is complicated and highly nonlinear and the set of independent generalized coordinates must be changed every time when the pivot of $\mathbf{C}_{\boldsymbol{q}_d}$ approaches zero.

2.4 Kinematic joint description

In this section, geometric constraint equations are derived for three basic constraint components, which can be further applied to the modeling of spherical joints, revolute joints, cylindrical joints and translational joints. The terms within the equations of motion that are related to the constraints are formulated so that they can easily be incorporated into multibody dynamics codes.

2.4.1 Basic Constraints

Joints in multibody systems can be described as combinations of three basic constraints. These basic constraints are the spherical constraint and two different perpendicularity constraint conditions. The basic constraints for rigid bodies have been presented e.g. in References [10] and [9]. For flexible bodies, however, there is no comprehensive analytic representation which could describe all of the components in Eq. (24) that are related to the constraints.

Spherical Constraint on Two Points

The spherical constraint on two points, which is depicted in Fig. 2, is a simple basic constraint that prevents translational movement between two bodies. The constraint equation can be defined at given points P^i and P^j. This basic constraint removes three degrees of freedom from the system.

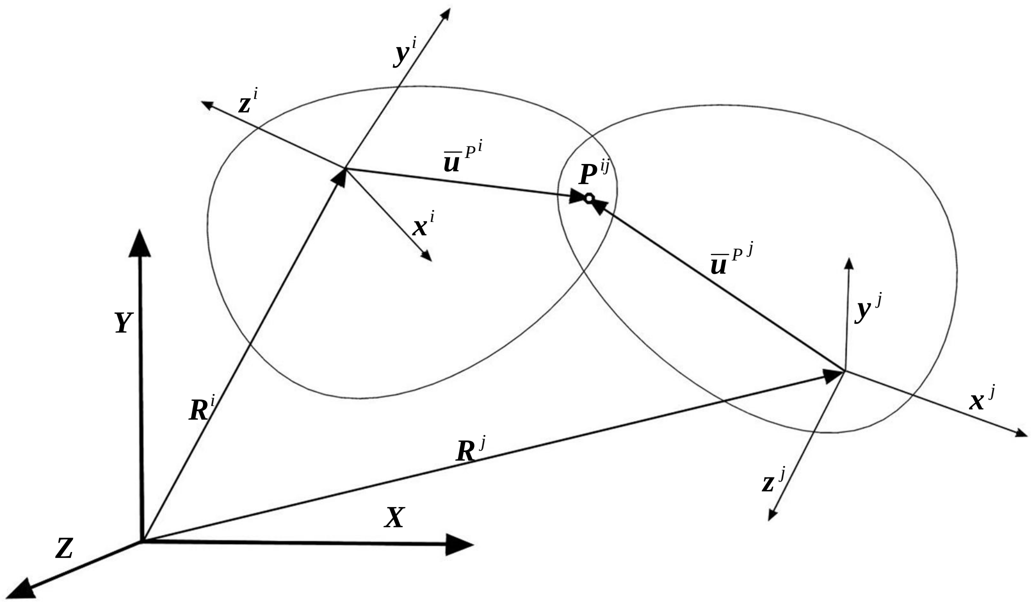


Figure 2. Spherical constraint on two points.

The constraint equation associated with points P^i and P^j can be written as follows:

$$\boldsymbol{C}^s = \boldsymbol{R}^j + \mathbf{A}^j\,\bar{\boldsymbol{u}}^{Pj} - \boldsymbol{R}^i - \mathbf{A}^i\,\bar{\boldsymbol{u}}^{Pi} = \boldsymbol{0}\,. \tag{45}$$

By differentiating Eq. (45) twice with respect to time, the following equation can be obtained:

$$\begin{aligned}\ddot{\boldsymbol{C}}^s &= \mathbf{C}^s_{\boldsymbol{q}}\,\ddot{\boldsymbol{q}} + \left(\mathbf{C}^s_{\boldsymbol{q}}\,\dot{\boldsymbol{q}}\right)_{\boldsymbol{q}}\dot{\boldsymbol{q}} \\ &= \ddot{\boldsymbol{R}}^j - \mathbf{A}^j\,\tilde{\bar{\boldsymbol{u}}}^{Pj}\dot{\bar{\boldsymbol{\omega}}}^j + \mathbf{A}^j\,\tilde{\bar{\boldsymbol{\omega}}}^j\tilde{\bar{\boldsymbol{\omega}}}^j\bar{\boldsymbol{u}}^{Pj} + 2\mathbf{A}^j\,\tilde{\bar{\boldsymbol{\omega}}}^j\dot{\bar{\boldsymbol{u}}}^{Pj} + \mathbf{A}^j\boldsymbol{\psi}^{Pj}_R\,\ddot{\boldsymbol{q}}^j_f \\ &\quad - \ddot{\boldsymbol{R}}^i + \mathbf{A}^i\,\tilde{\bar{\boldsymbol{u}}}^{Pi}\dot{\bar{\boldsymbol{\omega}}}^i - \mathbf{A}^i\,\tilde{\bar{\boldsymbol{\omega}}}^i\tilde{\bar{\boldsymbol{\omega}}}^i\bar{\boldsymbol{u}}^{Pi} - 2\mathbf{A}^i\,\tilde{\bar{\boldsymbol{\omega}}}^i\dot{\bar{\boldsymbol{u}}}^{Pi} - \mathbf{A}^i\boldsymbol{\psi}^{Pi}_R\,\ddot{\boldsymbol{q}}^i_f\,.\end{aligned} \tag{46}$$

Based on Eq. (46), the following terms can be obtained for generalized coordinates related to the translation, orientation and flexibility of the Jacobian matrix:

$$\mathbf{C}^s_{\boldsymbol{q}} = \begin{bmatrix} -\mathbf{I} & \mathbf{A}^i\,\tilde{\bar{\boldsymbol{u}}}^{Pi} & -\mathbf{A}^i\boldsymbol{\psi}^{Pi}_R & \mathbf{I} & -\mathbf{A}^j\,\tilde{\bar{\boldsymbol{u}}}^{Pj} & \mathbf{A}^j\boldsymbol{\psi}^{Pj}_R \end{bmatrix}. \tag{47}$$

Similarly, a vector that includes quadratic velocity terms can be obtained as follows:

$$\boldsymbol{Q}^{c^s} = -\left(\mathbf{C}_{\boldsymbol{q}}\,\dot{\boldsymbol{q}}\right)_{\boldsymbol{q}}\dot{\boldsymbol{q}} = -\mathbf{A}^j\,\tilde{\bar{\boldsymbol{\omega}}}^j\left(\tilde{\bar{\boldsymbol{\omega}}}^j\bar{\boldsymbol{u}}^{Pj} + 2\dot{\bar{\boldsymbol{u}}}^{Pj}\right) + \mathbf{A}^i\,\tilde{\bar{\boldsymbol{\omega}}}^i\left(\tilde{\bar{\boldsymbol{\omega}}}^i\bar{\boldsymbol{u}}^{Pi} + 2\dot{\bar{\boldsymbol{u}}}^{Pi}\right). \tag{48}$$

Perpendicular Constraint C^{d1}

The perpendicular constraint (type 1) preventing the rotation of vectors with respect to each other on levels which are not perpendicular to each other. The perpendicularity constraint is illustrated in Fig. 3. This basic constraint can be described with one constraint equation, which removes one degree of freedom from the system.

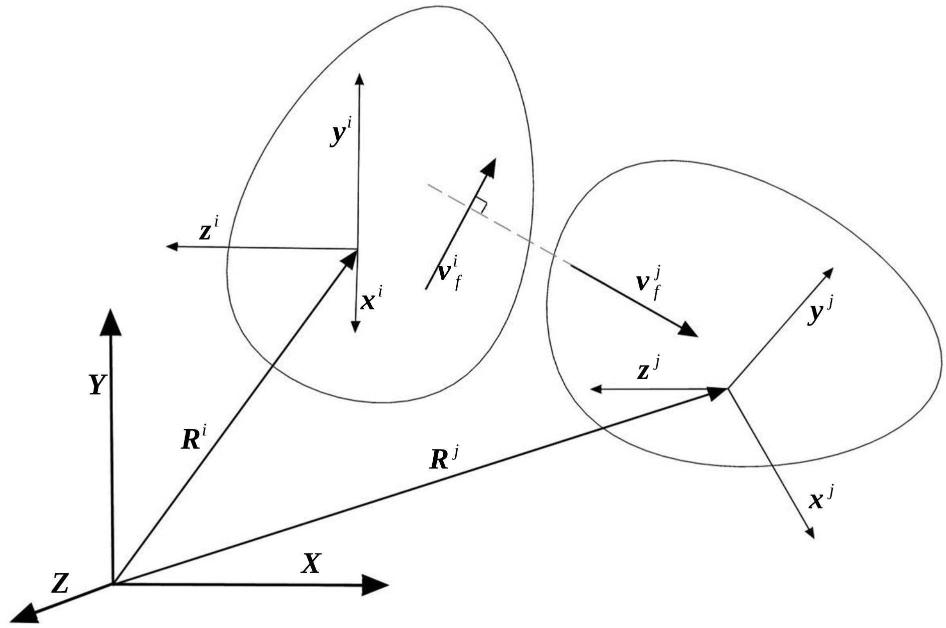


Figure 3. Type 1 perpendicular constraint.

The constraint equation for a perpendicular constraint of vectors can be written as

$$C^{d1} = \boldsymbol{v}_f^{i\,\mathrm{T}}\boldsymbol{v}_f^{j} = \bar{\boldsymbol{v}}_f^{i\,\mathrm{T}}\,\mathbf{A}^{i\,\mathrm{T}}\mathbf{A}^{j}\,\bar{\boldsymbol{v}}_f^{j} = \bar{\boldsymbol{v}}^{i\,\mathrm{T}}\,\mathbf{A}_f^{Pi\,\mathrm{T}}\,\mathbf{A}^{i\,\mathrm{T}}\mathbf{A}^{j}\mathbf{A}_f^{Pj}\,\bar{\boldsymbol{v}}^{j} = 0\,. \tag{49}$$

By differentiating the equation twice with respect to time, the following equation can be obtained:

$$\begin{aligned}
\ddot{C}^{d1} &= C_{\boldsymbol{q}}^{d1}\,\ddot{\boldsymbol{q}} + \left(C_{\boldsymbol{q}}^{d1}\,\dot{\boldsymbol{q}}\right)_{\boldsymbol{q}}\dot{\boldsymbol{q}} = \ddot{\boldsymbol{v}}_f^{i\,\mathrm{T}}\boldsymbol{v}_f^{j} + \boldsymbol{v}_f^{i\,\mathrm{T}}\ddot{\boldsymbol{v}}_f^{j} + 2\dot{\boldsymbol{v}}_f^{i\,\mathrm{T}}\dot{\boldsymbol{v}}_f^{j} \\
&= -\bar{\boldsymbol{v}}_f^{j\,\mathrm{T}}\,\mathbf{A}^{j\,\mathrm{T}}\mathbf{A}^{i}\,\tilde{\bar{\boldsymbol{v}}}_f^{i}\dot{\bar{\boldsymbol{\omega}}}^{i} - \bar{\boldsymbol{v}}_f^{j\,\mathrm{T}}\,\mathbf{A}^{j\,\mathrm{T}}\mathbf{A}^{i}\,\tilde{\bar{\boldsymbol{v}}}^{i}\,\boldsymbol{\psi}_\theta^{Pi}\,\ddot{\boldsymbol{q}}_f^{i} - \bar{\boldsymbol{v}}_f^{i\,\mathrm{T}}\,\mathbf{A}^{i\,\mathrm{T}}\mathbf{A}^{j}\,\tilde{\bar{\boldsymbol{v}}}_f^{j}\dot{\bar{\boldsymbol{\omega}}}^{j} \\
&\quad -\bar{\boldsymbol{v}}_f^{i\,\mathrm{T}}\,\mathbf{A}^{i\,\mathrm{T}}\mathbf{A}^{j}\,\tilde{\bar{\boldsymbol{v}}}^{j}\,\boldsymbol{\psi}_\theta^{Pj}\,\ddot{\boldsymbol{q}}_f^{j} \quad + \bar{\boldsymbol{v}}_f^{j\,\mathrm{T}}\,\mathbf{A}^{j\,\mathrm{T}}\mathbf{A}^{i}\,\tilde{\bar{\boldsymbol{\omega}}}^{i}\left(\tilde{\bar{\boldsymbol{\omega}}}^{i}\bar{\boldsymbol{v}}_f^{i} + 2\dot{\bar{\boldsymbol{v}}}_f^{i}\right) \\
&\quad + \bar{\boldsymbol{v}}_f^{i\,\mathrm{T}}\,\mathbf{A}^{i\,\mathrm{T}}\mathbf{A}^{j}\,\tilde{\bar{\boldsymbol{\omega}}}^{j}\left(\tilde{\bar{\boldsymbol{\omega}}}^{j}\bar{\boldsymbol{v}}_f^{j} + 2\dot{\bar{\boldsymbol{v}}}_f^{j}\right) + 2\left(\mathbf{A}^{i}\,\tilde{\bar{\boldsymbol{\omega}}}^{i}\bar{\boldsymbol{v}}_f^{i} + \mathbf{A}^{i}\,\dot{\bar{\boldsymbol{v}}}_f^{i}\right)^{\mathrm{T}}\left(\mathbf{A}^{j}\,\tilde{\bar{\boldsymbol{\omega}}}^{j}\bar{\boldsymbol{v}}_f^{j} + \mathbf{A}^{j}\,\dot{\bar{\boldsymbol{v}}}_f^{j}\right).
\end{aligned} \tag{50}$$

Based on Eq. (50), the following terms can be obtained for generalized coordinates related to the translation, orientation and flexibility of the Jacobian matrix:

$$\begin{aligned}
C_{\boldsymbol{q}}^{d1} = \Big[0 \quad & -\bar{\boldsymbol{v}}_f^{j\,\mathrm{T}}\,\mathbf{A}^{j\,\mathrm{T}}\mathbf{A}^{i}\,\tilde{\bar{\boldsymbol{v}}}_f^{i} \quad -\bar{\boldsymbol{v}}_f^{j\,\mathrm{T}}\,\mathbf{A}^{j\,\mathrm{T}}\mathbf{A}^{i}\,\tilde{\bar{\boldsymbol{v}}}^{i}\,\boldsymbol{\psi}_\theta^{Pi} \quad \cdots \\
& \cdots \quad 0 \quad -\bar{\boldsymbol{v}}_f^{i\,\mathrm{T}}\,\mathbf{A}^{i\,\mathrm{T}}\mathbf{A}^{j}\,\tilde{\bar{\boldsymbol{v}}}_f^{j} \quad -\bar{\boldsymbol{v}}_f^{i\,\mathrm{T}}\,\mathbf{A}^{i\,\mathrm{T}}\mathbf{A}^{j}\,\tilde{\bar{\boldsymbol{v}}}^{j}\,\boldsymbol{\psi}_\theta^{Pj}\Big]\,.
\end{aligned} \tag{51}$$

Correspondingly, the term that includes quadratic velocity terms can be represented as

$$\begin{aligned}
Q^{c\,d1} = -\left(\mathbf{C}_{\boldsymbol{q}}\,\dot{\boldsymbol{q}}\right)_{\boldsymbol{q}}\dot{\boldsymbol{q}} = -\bar{\boldsymbol{v}}_f^{j\,\mathrm{T}}\,\mathbf{A}^{j\,\mathrm{T}}\mathbf{A}^{i}\,\tilde{\bar{\boldsymbol{\omega}}}^{i}\left(\tilde{\bar{\boldsymbol{\omega}}}^{i}\bar{\boldsymbol{v}}_f^{i} + 2\dot{\bar{\boldsymbol{v}}}_f^{i}\right) - \bar{\boldsymbol{v}}_f^{i\,\mathrm{T}}\,\mathbf{A}^{i\,\mathrm{T}}\mathbf{A}^{j}\,\tilde{\bar{\boldsymbol{\omega}}}^{j}\left(\tilde{\bar{\boldsymbol{\omega}}}^{j}\bar{\boldsymbol{v}}_f^{j} + 2\dot{\bar{\boldsymbol{v}}}_f^{j}\right) \\
-2\left(\mathbf{A}^{i}\,\tilde{\bar{\boldsymbol{\omega}}}^{i}\bar{\boldsymbol{v}}_f^{i} + \mathbf{A}^{i}\,\dot{\bar{\boldsymbol{v}}}_f^{i}\right)^{\mathrm{T}}\left(\mathbf{A}^{j}\,\tilde{\bar{\boldsymbol{\omega}}}^{j}\bar{\boldsymbol{v}}_f^{j} + \mathbf{A}^{j}\,\dot{\bar{\boldsymbol{v}}}_f^{j}\right).
\end{aligned} \tag{52}$$

Perpendicular Constraint C^{d2}

Perpendicular constraint type 2 differs from type 1 in that one of the vectors is defined as constant with respect to body *i*, whereas the other is defined between the bodies as shown in Fig. 4. This constraint is also known as the point on plane since it contains one constraint equation eliminating one degree of freedom.

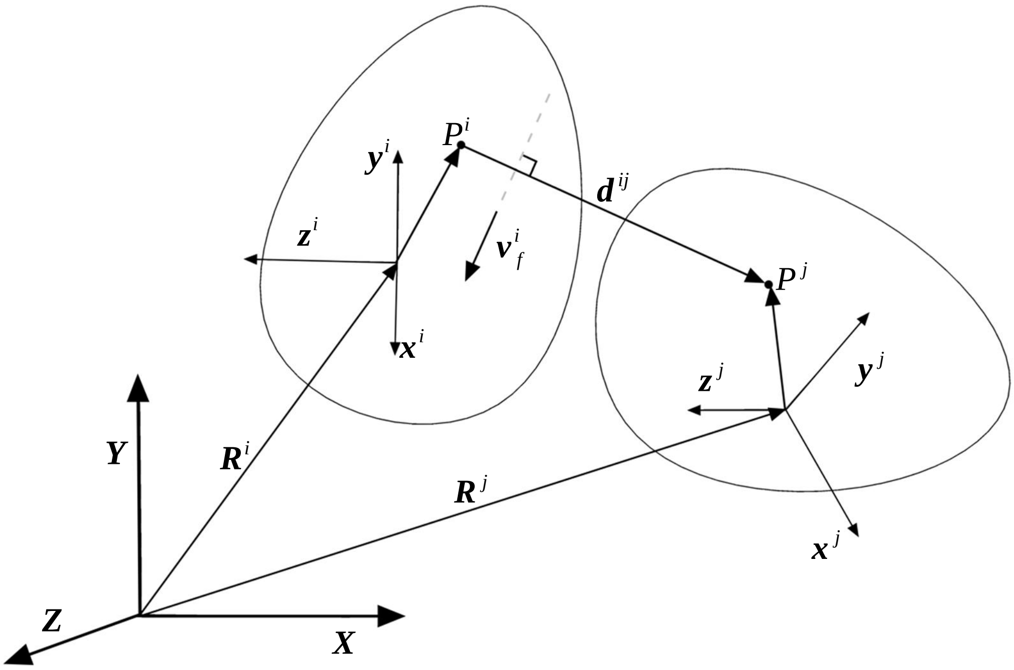


Figure 4. Type 2 perpendicularity constraint.

The constraint equation for a type 2 perpendicularity constraint can be represented as

$$\begin{aligned} C^{d2} &= \boldsymbol{v}_f^{i\,\mathrm{T}} \boldsymbol{d}^{ij} = \bar{\boldsymbol{v}}_f^{i\,\mathrm{T}} \mathbf{A}^{i\mathrm{T}} \left(\boldsymbol{R}^j + \mathbf{A}^j \bar{\boldsymbol{u}}^{Pj} - \boldsymbol{R}^i - \mathbf{A}^i \bar{\boldsymbol{u}}^{Pi} \right) \\ &= \bar{\boldsymbol{v}}_f^{i\,\mathrm{T}} \mathbf{A}_f^{Pi\,\mathrm{T}} \mathbf{A}^{i\mathrm{T}} \left(\boldsymbol{R}^j + \mathbf{A}^j \bar{\boldsymbol{u}}^{Pj} - \boldsymbol{R}^i - \mathbf{A}^i \bar{\boldsymbol{u}}^{Pi} \right) = 0, \end{aligned} \tag{53}$$

where $\boldsymbol{d}^{ij}$ is vector from P^i to P^j defined in the global coordinate system. By differentiating the equation twice with respect to time, the following equation can be obtained:

$$
\begin{aligned}
\ddot{C}^{d2} &= C^{d2}_{\boldsymbol{q}}\ddot{\boldsymbol{q}} + \left(C^{d2}_{\boldsymbol{q}}\dot{\boldsymbol{q}}\right)_{\boldsymbol{q}}\dot{\boldsymbol{q}} = \ddot{\boldsymbol{v}}_f^{i\,\mathrm{T}}\boldsymbol{d}^{ij} + \boldsymbol{v}_f^{i\,\mathrm{T}}\ddot{\boldsymbol{d}}^{ij} + 2\dot{\boldsymbol{v}}_f^{i\,\mathrm{T}}\dot{\boldsymbol{d}}^{ij} \\
&= -\bar{\boldsymbol{v}}_f^{i\,\mathrm{T}}\mathbf{A}^{i\mathrm{T}}\ddot{\boldsymbol{R}}^i + \left(\bar{\boldsymbol{v}}_f^{i\,\mathrm{T}}\tilde{\bar{\boldsymbol{u}}}^{Pi} - \boldsymbol{d}^{ij\mathrm{T}}\mathbf{A}^i\tilde{\bar{\boldsymbol{v}}}_f^i\right)\dot{\bar{\boldsymbol{\omega}}}^i - \left(\bar{\boldsymbol{v}}_f^{i\,\mathrm{T}}\boldsymbol{\psi}_R^{Pi} + \boldsymbol{d}^{ij\mathrm{T}}\mathbf{A}^i\tilde{\bar{\boldsymbol{v}}}^i\boldsymbol{\psi}_\theta^{Pi}\right)\ddot{\boldsymbol{q}}_f^i \\
&\quad + \bar{\boldsymbol{v}}_f^{i\,\mathrm{T}}\mathbf{A}^{i\mathrm{T}}\ddot{\boldsymbol{R}}^j - \bar{\boldsymbol{v}}_f^{i\,\mathrm{T}}\mathbf{A}^{i\mathrm{T}}\mathbf{A}^j\tilde{\bar{\boldsymbol{u}}}^{Pj}\dot{\bar{\boldsymbol{\omega}}}^j + \bar{\boldsymbol{v}}_f^{i\,\mathrm{T}}\mathbf{A}^{i\mathrm{T}}\mathbf{A}^j\boldsymbol{\psi}_R^{Pj}\ddot{\boldsymbol{q}}_f^j \\
&\quad + \boldsymbol{d}^{ij\mathrm{T}}\mathbf{A}^i\tilde{\bar{\boldsymbol{\omega}}}^i\left(\tilde{\bar{\boldsymbol{\omega}}}^i\bar{\boldsymbol{v}}_f^i + 2\dot{\bar{\boldsymbol{v}}}_f^i\right) + \bar{\boldsymbol{v}}_f^{i\,\mathrm{T}}\mathbf{A}^{i\mathrm{T}}\mathbf{A}^j\tilde{\bar{\boldsymbol{\omega}}}^j\left(\tilde{\bar{\boldsymbol{\omega}}}^j\bar{\boldsymbol{u}}^{Pj} + 2\dot{\bar{\boldsymbol{u}}}^{Pj}\right) \\
&\quad + 2\left(\mathbf{A}^i\tilde{\bar{\boldsymbol{\omega}}}^i\bar{\boldsymbol{v}}_f^i + \mathbf{A}^i\tilde{\dot{\bar{\boldsymbol{\varepsilon}}}}^{Pi}\bar{\boldsymbol{v}}^i\right)^{\mathrm{T}}\left(\dot{\boldsymbol{R}}^j + \mathbf{A}^j\tilde{\bar{\boldsymbol{\omega}}}^j\bar{\boldsymbol{u}}^{Pj} + \mathbf{A}^j\dot{\bar{\boldsymbol{u}}}^{Pj} - \dot{\boldsymbol{R}}^i\right) \\
&\quad + \bar{\boldsymbol{v}}_f^{i\,\mathrm{T}}\tilde{\bar{\boldsymbol{\omega}}}^i\tilde{\bar{\boldsymbol{\omega}}}^i\bar{\boldsymbol{u}}^{Pi} + 2\bar{\boldsymbol{v}}_f^{i\,\mathrm{T}}\tilde{\dot{\bar{\boldsymbol{\varepsilon}}}}^{Pi}\tilde{\bar{\boldsymbol{\omega}}}^i\bar{\boldsymbol{u}}^{Pi} + 2\bar{\boldsymbol{v}}_f^{i\,\mathrm{T}}\tilde{\dot{\bar{\boldsymbol{\varepsilon}}}}^{Pi}\dot{\bar{\boldsymbol{u}}}^{Pi}\,.
\end{aligned}
\tag{54}
$$

Based on Eq. (54), the following terms are obtained for generalized coordinates related to the translation, orientation and flexibility of the Jacobian matrix:

$$
\begin{aligned}
C^{d2}_{\boldsymbol{q}} = \Big[& -\bar{\boldsymbol{v}}_f^{i\,\mathrm{T}}\mathbf{A}^{i\mathrm{T}} \quad \bar{\boldsymbol{v}}_f^{i\,\mathrm{T}}\tilde{\bar{\boldsymbol{u}}}^{Pi} - \boldsymbol{d}^{ij\mathrm{T}}\mathbf{A}^i\tilde{\bar{\boldsymbol{v}}}_f^i \quad -\bar{\boldsymbol{v}}_f^{i\,\mathrm{T}}\boldsymbol{\psi}_R^{Pi} - \boldsymbol{d}^{ij\mathrm{T}}\mathbf{A}^i\tilde{\bar{\boldsymbol{v}}}^i\boldsymbol{\psi}_\theta^{Pi} \quad \cdots \\
& \cdots \quad \bar{\boldsymbol{v}}_f^{i\,\mathrm{T}}\mathbf{A}^{i\mathrm{T}} \quad -\bar{\boldsymbol{v}}_f^{i\,\mathrm{T}}\mathbf{A}^{i\mathrm{T}}\mathbf{A}^j\tilde{\bar{\boldsymbol{u}}}^{Pj} \quad \bar{\boldsymbol{v}}_f^{i\,\mathrm{T}}\mathbf{A}^{i\mathrm{T}}\mathbf{A}^j\boldsymbol{\psi}_R^{Pj}\Big].
\end{aligned}
\tag{55}
$$

Correspondingly, the term that contains quadratic velocity terms can be expressed as follows:

$$
\begin{aligned}
Q^{c\,d2} = -\left(\mathbf{C}_{\boldsymbol{q}}\dot{\boldsymbol{q}}\right)_{\boldsymbol{q}}\dot{\boldsymbol{q}} &= -\boldsymbol{d}^{ij\mathrm{T}}\mathbf{A}^i\tilde{\bar{\boldsymbol{\omega}}}^i\left(\tilde{\bar{\boldsymbol{\omega}}}^i\bar{\boldsymbol{v}}_f^i + 2\dot{\bar{\boldsymbol{v}}}_f^i\right) - \bar{\boldsymbol{v}}_f^{i\,\mathrm{T}}\mathbf{A}^{i\mathrm{T}}\mathbf{A}^j\tilde{\bar{\boldsymbol{\omega}}}^j\left(\tilde{\bar{\boldsymbol{\omega}}}^j\bar{\boldsymbol{u}}^{Pj} + 2\dot{\bar{\boldsymbol{u}}}^{Pj}\right) \\
&\quad - 2\left(\mathbf{A}^i\tilde{\bar{\boldsymbol{\omega}}}^i\bar{\boldsymbol{v}}_f^i + \mathbf{A}^i\dot{\bar{\boldsymbol{v}}}_f^i\right)^{\mathrm{T}}\left(\dot{\boldsymbol{R}}^j + \mathbf{A}^j\tilde{\bar{\boldsymbol{\omega}}}^j\bar{\boldsymbol{u}}^{Pj} + \mathbf{A}^j\dot{\bar{\boldsymbol{u}}}^{Pj} - \dot{\boldsymbol{R}}^i\right) \\
&\quad - \bar{\boldsymbol{v}}_f^{i\,\mathrm{T}}\tilde{\bar{\boldsymbol{\omega}}}^i\tilde{\bar{\boldsymbol{\omega}}}^i\bar{\boldsymbol{u}}^{Pi} + 2\dot{\bar{\boldsymbol{v}}}_f^{i\,\mathrm{T}}\tilde{\bar{\boldsymbol{\omega}}}^i\bar{\boldsymbol{u}}^{Pi} + 2\dot{\bar{\boldsymbol{v}}}_f^{i\,\mathrm{T}}\dot{\bar{\boldsymbol{u}}}^{Pi}\,.
\end{aligned}
\tag{56}
$$

2.4.2 Modeling of Joints Based on Basic Constraints

In this section, the basic joint types used in multibody dynamics modeling applying the basic constraints presented above are briefly introduced. With different combinations of basic constraints, it is possible to model any joint. Table 1 summarizes partial derivatives with regard to generalized coordinates for each basic constraint.

Table 1. Partial derivatives for basic constraints.

	$\boldsymbol{C}^s$	C^{d1}	C^{d2}
$\mathbf{C}_{\boldsymbol{R}i}$	$-\mathbf{I}$	$\mathbf{0}$	$-\bar{\boldsymbol{v}}_f^{i\,\mathrm{T}}\mathbf{A}^{i\mathrm{T}}$
$\mathbf{C}_{\bar{\theta}i}$	$\mathbf{A}^i\tilde{\bar{\boldsymbol{u}}}^{Pi}$	$-\bar{\boldsymbol{v}}_f^{j\,\mathrm{T}}\mathbf{A}^{j\mathrm{T}}\mathbf{A}^i\tilde{\bar{\boldsymbol{v}}}_f^i$	$\bar{\boldsymbol{v}}_f^{i\,\mathrm{T}}\tilde{\bar{\boldsymbol{u}}}^{Pi} - \boldsymbol{d}^{ij\mathrm{T}}\mathbf{A}^i\tilde{\bar{\boldsymbol{v}}}_f^i$
$\mathbf{C}_{\boldsymbol{q}_f^i}$	$-\mathbf{A}^i\boldsymbol{\psi}_R^{Pi}$	$-\bar{\boldsymbol{v}}_f^{j\,\mathrm{T}}\mathbf{A}^{j\mathrm{T}}\mathbf{A}^i\tilde{\bar{\boldsymbol{v}}}^i\boldsymbol{\psi}_\theta^{Pi}$	$-\bar{\boldsymbol{v}}_f^{i\,\mathrm{T}}\boldsymbol{\psi}_R^{Pi} - \boldsymbol{d}^{ij\mathrm{T}}\mathbf{A}^i\tilde{\bar{\boldsymbol{v}}}^i\boldsymbol{\psi}_\theta^{Pi}$
$\mathbf{C}_{\boldsymbol{R}j}$	$\mathbf{I}$	$\mathbf{0}$	$\bar{\boldsymbol{v}}_f^{i\,\mathrm{T}}\mathbf{A}^{i\mathrm{T}}$

$\mathbf{C}_{\bar{\theta}^j}$	$-\mathbf{A}^j\,\tilde{\bar{\boldsymbol{u}}}^{Pj}$	$-\bar{\boldsymbol{v}}_f^{i\,\mathrm{T}}\,\mathbf{A}^{i\mathrm{T}}\mathbf{A}^j\,\tilde{\bar{\boldsymbol{v}}}_f^j$	$-\bar{\boldsymbol{v}}_f^{i\,\mathrm{T}}\,\mathbf{A}^{i\mathrm{T}}\mathbf{A}^j\,\tilde{\bar{\boldsymbol{u}}}^{Pj}$
$\mathbf{C}_{\boldsymbol{q}_f^i}$	$\mathbf{A}^j\boldsymbol{\psi}_R^{Pj}$	$-\bar{\boldsymbol{v}}_f^{i\,\mathrm{T}}\,\mathbf{A}^{i\mathrm{T}}\mathbf{A}^j\,\tilde{\bar{\boldsymbol{v}}}^j\,\boldsymbol{\psi}_\theta^{Pj}$	$\bar{\boldsymbol{v}}_f^{i\,\mathrm{T}}\,\mathbf{A}^{i\mathrm{T}}\mathbf{A}^j\boldsymbol{\psi}_R^{Pj}$

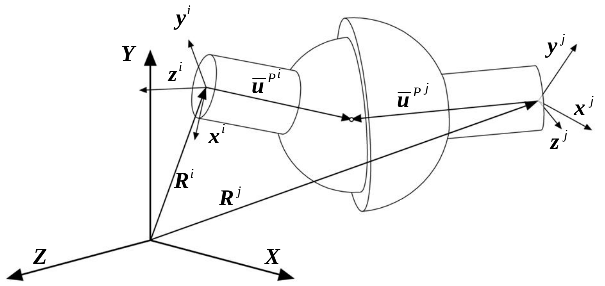
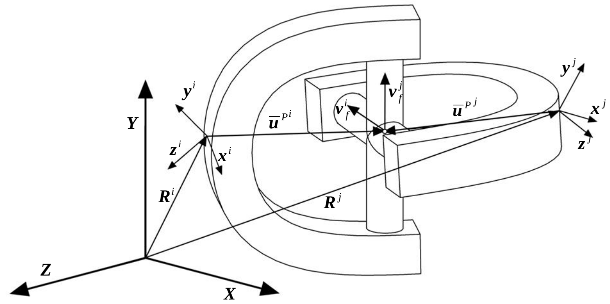
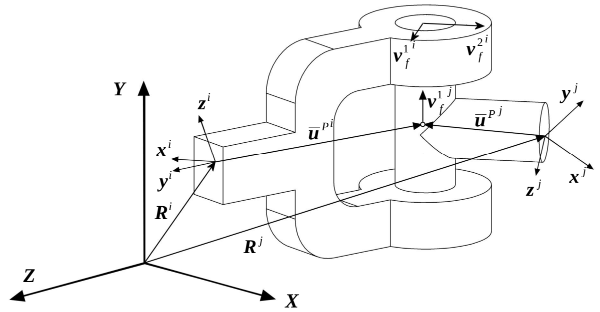
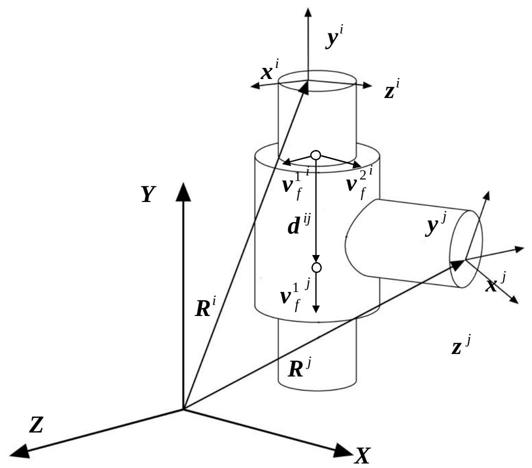
Table *2* presents the components of the constraint force vector related to the basic constraints.

Table 2. Components of the constraint force vector related to basic constraints.

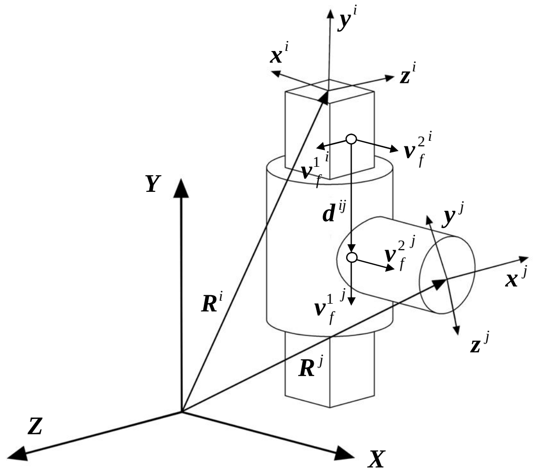
	$\boldsymbol{Q}_c$
$\boldsymbol{Q}^{c^s}$	$-\mathbf{A}^j\,\tilde{\bar{\boldsymbol{\omega}}}^j\left(\tilde{\bar{\boldsymbol{\omega}}}^j\bar{\boldsymbol{u}}^{Pj}+2\dot{\bar{\boldsymbol{u}}}^{Pj}\right)+\mathbf{A}^i\,\tilde{\bar{\boldsymbol{\omega}}}^i\left(\tilde{\bar{\boldsymbol{\omega}}}^i\bar{\boldsymbol{u}}^{Pi}+2\dot{\bar{\boldsymbol{u}}}^{Pi}\right)$
$Q^{c^{d1}}$	$-\bar{\boldsymbol{v}}_f^{j\,\mathrm{T}}\,\mathbf{A}^{j\mathrm{T}}\mathbf{A}^i\,\tilde{\bar{\boldsymbol{\omega}}}^i\left(\tilde{\bar{\boldsymbol{\omega}}}^i\bar{\boldsymbol{v}}_f^i+2\dot{\bar{\boldsymbol{v}}}_f^i\right)-\bar{\boldsymbol{v}}_f^{i\,\mathrm{T}}\,\mathbf{A}^{i\mathrm{T}}\mathbf{A}^j\,\tilde{\bar{\boldsymbol{\omega}}}^j\left(\tilde{\bar{\boldsymbol{\omega}}}^j\bar{\boldsymbol{v}}_f^j+2\dot{\bar{\boldsymbol{v}}}_f^j\right)$ $-2\left(\mathbf{A}^i\,\tilde{\bar{\boldsymbol{\omega}}}^i\bar{\boldsymbol{v}}_f^i+\mathbf{A}^i\,\dot{\bar{\boldsymbol{v}}}_f^i\right)^{\mathrm{T}}\left(\mathbf{A}^j\,\tilde{\bar{\boldsymbol{\omega}}}^j\bar{\boldsymbol{v}}_f^j+\mathbf{A}^j\,\dot{\bar{\boldsymbol{v}}}_f^j\right)$
$Q^{c^{d2}}$	$-\boldsymbol{d}^{ij\,\mathrm{T}}\,\mathbf{A}^i\,\tilde{\bar{\boldsymbol{\omega}}}^i\left(\tilde{\bar{\boldsymbol{\omega}}}^i\bar{\boldsymbol{v}}_f^i+2\dot{\bar{\boldsymbol{v}}}_f^i\right)-\bar{\boldsymbol{v}}_f^{i\,\mathrm{T}}\,\mathbf{A}^{i\mathrm{T}}\mathbf{A}^j\,\tilde{\bar{\boldsymbol{\omega}}}^j\left(\tilde{\bar{\boldsymbol{\omega}}}^j\bar{\boldsymbol{u}}^{Pj}+2\dot{\bar{\boldsymbol{u}}}^{Pj}\right)$ $-2\left(\mathbf{A}^i\,\tilde{\bar{\boldsymbol{\omega}}}^i\bar{\boldsymbol{v}}_f^i+\mathbf{A}^i\,\dot{\bar{\boldsymbol{v}}}_f^i\right)^{\mathrm{T}}\left(\dot{\boldsymbol{R}}^j+\mathbf{A}^j\,\tilde{\bar{\boldsymbol{\omega}}}^j\bar{\boldsymbol{u}}^{Pj}+\mathbf{A}^j\,\dot{\bar{\boldsymbol{u}}}^{Pj}-\dot{\boldsymbol{R}}^i\right)$ $-\bar{\boldsymbol{v}}_f^{i\,\mathrm{T}}\tilde{\bar{\boldsymbol{\omega}}}^i\tilde{\bar{\boldsymbol{\omega}}}^i\bar{\boldsymbol{u}}^{Pi}+2\dot{\bar{\boldsymbol{v}}}_f^{i\,\mathrm{T}}\tilde{\bar{\boldsymbol{\omega}}}^i\bar{\boldsymbol{u}}^{Pi}+2\dot{\bar{\boldsymbol{v}}}_f^{i\,\mathrm{T}}\dot{\bar{\boldsymbol{u}}}^{Pi}$

In the case of spherical joints, universal joints and revolute joints, the constraint location remains in place and the joints can be modeled by changing the constraints of the rotations. Joints such as cylinder and translational joints that enable relative translational movement between bodies are challenging to model due to their varying location, to which the constraint is applied. For flexible bodies, the location of the constraint must be solved for each time step. The location can be found for instance by applying interpolation between the nodes of the joint. Table 3 summarizes the descriptions of the joints and constraint equations with which they can be modeled.

Table 3. Descriptions of joints and basic constraint equations applied to them.

Joint	Illustration	Constraint equations
Spherical joint The spherical joint is the simplest to model and can be described using one of the basic constraints. In the point constraint, the global coordinates of the points must be located overlappingly. The spherical joint constrains three degrees of freedom from the system.		C^{s}
Universal joint Universal joints can be modeled with a spherical constraint and a type 1 perpendicular constraint. The universal joint removes four degrees of freedom from the system.		C^{s} C^{d1}
Revolute joint A revolute joint can be adapted from the universal joint by adding another type 1 perpendicularity constraint. The revolute joint removes five degrees of freedom from the system.		C^{s} C^{d1} C^{d1}
Cylindrical joint The modeling of a cylindrical joint requires two type 1 perpendicular constraints to prevent the relative rotation of the bodies at point P^{j}, and two type 2 perpendicular constraints to maintain point P^{i} on the translational axis. The cylindrical joint removes four degrees of freedom from the system.		C^{d1} C^{d1} C^{d2} C^{d2}

<table>
<tr>
<td>Translational joint

A translational joint can be derived directly from the cylindrical joints by adding a type 1 perpendicularity constraint. A translational joint removes five degrees of freedom from the system.</td>
<td>
</td>
<td>C^{d1}
C^{d1}
C^{d1}
C^{d2}
C^{d2}</td>
</tr>
</table>

It is important to note that basic constraints can be combined in various other ways than the ones described in Table *3*. Due this fact, also more unconventional joints can be described.

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