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A Unified Formulation for Dimensional Synthesis of Stephenson Linkages

The unified formulation of dimensional synthesis of Stephenson linkages for motion generation is the subject of this paper. Burmester theory is applied to the six-bar linkage, which leads to a unified formulation applicable for all three types of Stephenson linkages. This is made possible by virtue of parameterized position vectors, which simplify the formulation of synthesis equations. A design example is included to demonstrate the application of the method developed. [DOI: 10.1115/1.4032701]

Keywords: six-bar linkage motion synthesis, Stephenson linkages, uniform synthesis equations, synthesis with incomplete data set

1 Introduction

A six-bar linkage as an one degrees-of-freedom mechanism can be used for motion generation, the same capability as four-bar linkages. Compared with the latter, the six-bar linkages in general offer a high synthesis flexibility. This means that a six-bar linkage can be designed with constraints on link dimensions and pivoting locations [1].

Some methods of synthesis of six-bar linkages are available in literatures. An early work on six-bar linkage synthesis can be found in Ref. [2], in which three mechanisms, a Watt linkage and two inversions of the Stephenson linkage, were considered. Design equations for Watt I and Stephenson I, II, and III path generators can be found in Ref. [3], which were obtained by extension of synthesis of a planar four-bar mechanism. A method of six-bar linkage synthesis was developed by Soh and McCarthy, where they considered the linkages as constrained 3R chains [4]. Many works were also reported in literatures on the solving of synthesis equations. An optimum synthesis method for six-bar linkages using differential evolution was reported in Ref. [5]. Dimensional synthesis of six-bar linkages was studied for a symmetrical Watt mechanism in Ref. [6]. An approach of synthesis by adaptive fitting was studied in Ref. [7]. Path synthesis of six-bar linkage by applying genetic algorithms was studied in Ref. [8]. Application of differential evolution algorithms in the design of six-bar linkages for path generation can be found in Ref. [9]. Function generation with a large number of separated precision points was studied in Ref. [10]. Moreover, problem of dimensional synthesis of Stephenson-II function generators without order, circuit, and branch defects was addressed in Ref. [11]. Mirth and Chase developed a method of circuit rectification for four-point precision synthesis of Stephenson six-bar linkages [12]. Ting and Dou studied the rotatability of any Stephenson six-bar linkage and developed an algorithm to identify its branch condition [13].

The subject of this study is the motion generation synthesis of the Stephenson linkages. A Stephenson linkage comprises two ternary links that are connected by binary links. Depending on the grounded link, either one of the two ternary links or a binary link, the Stephenson linkages are classified as three types, namely, Stephenson-I, II, and III linkages. The Stephenson linkages have two kinematic loops, the number of links in each loop varying with the linkage type. In most reported works, synthesis equations were formulated on closure equations of loops [14]. As the loop-closure equations depend on the topology of the linkages, the formulations have to be developed separately for each of the three types of Stephenson linkages. A unified formulation for Stephenson linkages is desirable.

In this paper, the dimensional synthesis of Stephenson linkages with a unified formulation is studied. Burmester theory is applied to the six-bar linkage, which leads to a unified formulation applicable for all three types of Stephenson linkages. A method of coordinate parameterization is developed for ternary links, which simplifies the formulation of synthesis equations. The method is demonstrated with a design example.

2 Problem Formulation

We start the formulation with the Burmester theory. The theory has been extensively studied, mainly for four-bar linkages including both planar [15,16] and spherical four-bar linkages [17,18]. In this work, we extend the classic Burmester problem from the four-bar linkage to the six-bar linkage. The Burmester problem in this context reads: A rigid body, as shown in Fig. 1, is to be guided through a discrete set of m poses, given by $\{\mathbf{r}_j, \psi_j\}_0^m$, where $\mathbf{r}_j$ is the position vector of a landmark point G of the body at the jth pose and ψ_j is the corresponding angle that a line of the body makes with a line of the frame. The problem consists in finding the joint centers A and B, a.k.a. the circlepoint and the

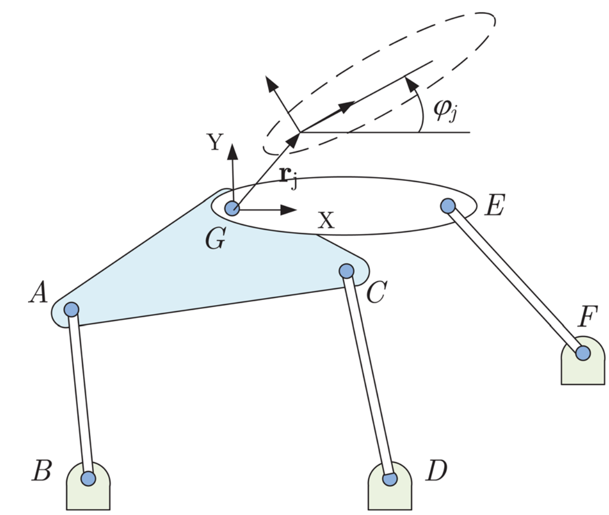


Fig. 1 A Stephenson-III linkage. Three dyads, namely, dyads *AB*, *CD*, and *EF* are to be synthesized.

Manuscript received October 20, 2015; final manuscript received February 2, 2016; published online March 7, 2016. Assoc. Editor: Raffaele Di Gregorio.

centerpoint, which define the *BA dyad*. Dyads *DC* and *FE* are determined likewise.

The problem at hand consists of finding the three revolute–revolute (RR) dyads to construct a six-bar linkage able to visit a set of prescribed poses of link *GE*. The linkage in Fig. 1 is of the Stephenson-III type, which is studied first. The other two will be considered in Sec. 4.

For the linkage synthesis as illustrated above, the origin of the coordinate system is taken at point *G*. The unknowns are the coordinates of points $A, B, \ldots, F$, expressed by vectors $\mathbf{a}, \mathbf{b}, \ldots, \mathbf{f}$. For a moving point, a subscript is attached to indicate the pose number. Moreover, we let $\mathbf{Q}_j$ and $\mathbf{R}_j$, $j = 1, \ldots, m$ be the rotation matrices of links *AG* and *GE*, respectively, with $\mathbf{Q}_0 = \mathbf{R}_0 = \mathbf{1}$.

3 Synthesis of Stephenson-III Linkage

Without loss of generality, we start the synthesis with a single RR dyad, as shown in Fig. 2. Under the usual rigid-body assumption, the synthesis equation is readily derived

$$||\underbrace{(\mathbf{r}_j - \mathbf{b}) + \mathbf{Q}_j\mathbf{a}_0}_{\mathbf{a}_j - \mathbf{b}}||^2 = ||\mathbf{a}_0 - \mathbf{b}||^2, \quad \text{for}\, j = 1, \ldots, m \tag{1}$$

where $\mathbf{a}_0$ and $\mathbf{b}$ are the position vectors of points A_0 and B, the design parameters of the linkage. $\mathbf{Q}_j$ denotes the rotation matrix carrying link *AG* from pose 0 to pose *j* by an angle $\phi_j = \theta_j - \theta_0$, as demonstrated in Fig. 3

Equation (1) is the vector form of the Burmester theory, which implies that the trajectory of point *A* is a circle of radius $r = ||\mathbf{a}_0 - \mathbf{b}||$, centered at point *B*.

Upon expansion of Eq. (1) and simplification, we obtain

$$\mathbf{b}^{\mathrm{T}}(\mathbf{1} - \mathbf{Q}_j)\mathbf{a}_0 + \mathbf{r}_j^{\mathrm{T}}\mathbf{Q}_j\mathbf{a}_0 - \mathbf{r}_j^{\mathrm{T}}\mathbf{b} + \frac{\mathbf{r}_j^{\mathrm{T}}\mathbf{r}_j}{2} = 0, \quad j = 1, \ldots, m \tag{2}$$

which are the *synthesis equations* that can be used to compute the design parameters, namely, $\mathbf{a}_0$ and $\mathbf{b}$. The equation contains four variables, which are the coordinates of points A_0 and B. The dyad admits exact solutions for at most five poses ($m = 4$), a well-known result [19].

Likewise, the synthesis equations for the RR dyad *CD* in Fig. 1 are

$$\mathbf{d}^{\mathrm{T}}(\mathbf{1} - \mathbf{Q}_j)\mathbf{c}_0 + \mathbf{r}_j^{\mathrm{T}}\mathbf{Q}_j\mathbf{c}_0 - \mathbf{r}_j^{\mathrm{T}}\mathbf{d} + \frac{1}{2}\mathbf{r}_j^{\mathrm{T}}\mathbf{r}_j = 0, \quad j = 1, \ldots, m \tag{3}$$

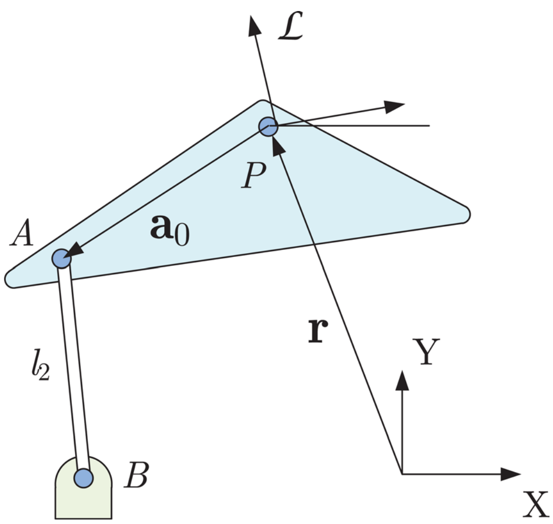


Fig. 2 A RR dyad

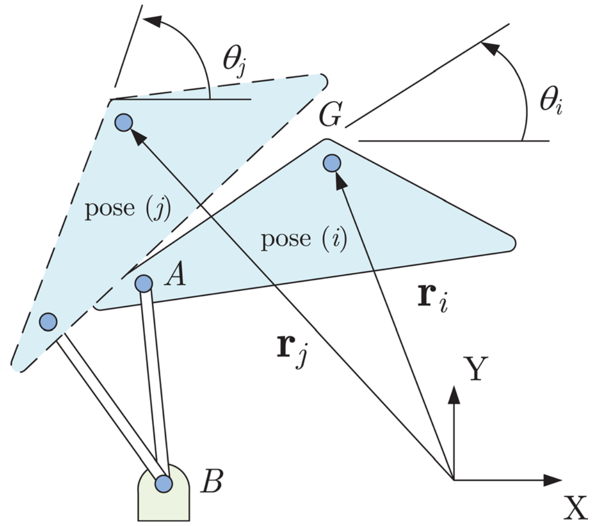


Fig. 3 Two separated poses of link *AG*

while the synthesis equations for the RR dyad *EF* are

$$\mathbf{f}^{\mathrm{T}}(\mathbf{1} - \mathbf{R}_j)\mathbf{e}_0 + \mathbf{r}_j^{\mathrm{T}}\mathbf{R}_j\mathbf{e}_0 - \mathbf{r}_j^{\mathrm{T}}\mathbf{f} + \frac{1}{2}\mathbf{r}_j^{\mathrm{T}}\mathbf{r}_j = 0, \quad j = 1, \ldots, m \tag{4}$$

where $\mathbf{R}_j$ denotes the orientation matrix of an angle ψ_j for the link *GE*, as shown in Fig. 1.

Equations (2)–(4) compose the set of synthesis equations for the Stephenson-III linkage, where variables include coordinates of points A_0, B, C_0, D, E_0, F, and orientation variable ϕ_j. The latter is a configuration-dependent variable, which is desirably eliminated.

3.1 Elimination of Motion Variables. The orientation angle ϕ_j of link *AG* is an intermediate motion variable that is desirably eliminated. This can be accomplished by utilizing synthesis equations of dyads *AB* and *CD*.

To begin with, all terms of Eq. (2) are expressed by writing $\mathbf{Q}_j$ in the form $\mathbf{Q}_j = c_j\mathbf{1} + s_j\mathbf{E}$, in which $s_j \equiv \sin\phi_j$ and $c_j \equiv \cos\phi_j$. Moreover, matrix 1 is the 2×2 identity matrix, while $\mathbf{E}$ is the matrix of rotation through an angle of 90 deg. Hence,

$$\begin{aligned}\mathbf{b}^{\mathrm{T}}(\mathbf{1} - \mathbf{Q}_j)\mathbf{a}_0 &= \mathbf{b}^{\mathrm{T}}(\mathbf{1} - c_j\mathbf{1} - s_j\mathbf{E})\mathbf{a}_0 \\ &= \mathbf{b}^{\mathrm{T}}\mathbf{a}_0 - c_j\mathbf{b}^{\mathrm{T}}\mathbf{a}_0 - s_j\mathbf{b}^{\mathrm{T}}\mathbf{E}\mathbf{a}_0\end{aligned} \tag{5a}$$

$$\begin{aligned}\mathbf{r}_j^{\mathrm{T}}\mathbf{Q}_j\mathbf{a}_0 &= \mathbf{r}_j^{\mathrm{T}}(c_j\mathbf{1} + s_j\mathbf{E})\mathbf{a}_0 \\ &= c_j\mathbf{r}_j^{\mathrm{T}}\mathbf{a}_0 + s_j\mathbf{r}_j^{\mathrm{T}}\mathbf{E}\mathbf{a}_0\end{aligned} \tag{5b}$$

Equation (2) can be written for dyad *AB* in an abbreviated form

$$A_1c_j + B_1s_j + C_1 = 0 \tag{6a}$$

with coefficients defined as

$$A_1 = \mathbf{r}_j^{\mathrm{T}}\mathbf{a}_0 - \mathbf{b}^{\mathrm{T}}\mathbf{a}_0 \tag{6b}$$

$$B_1 = \mathbf{r}_j^{\mathrm{T}}\mathbf{E}\mathbf{a}_0 - \mathbf{b}^{\mathrm{T}}\mathbf{E}\mathbf{a}_0 \tag{6c}$$

$$C_1 = \mathbf{b}^{\mathrm{T}}\mathbf{a}_0 - \mathbf{r}_j^{\mathrm{T}}\mathbf{b} + \mathbf{r}_j^{\mathrm{T}}\mathbf{r}_j/2 \tag{6d}$$

Likewise, the synthesis equation for dyad *CD* leads to

$$A_2c_j + B_2s_j + C_2 = 0 \tag{7a}$$

with

$$A_2 = \mathbf{r}_j^{\mathrm{T}} \mathbf{c}_0 - \mathbf{d}^{\mathrm{T}} \mathbf{c}_0 \tag{7b}$$

$$B_2 = \mathbf{r}_j^{\mathrm{T}} \mathbf{E} \mathbf{c}_0 - \mathbf{d}^{\mathrm{T}} \mathbf{E} \mathbf{c}_0 \tag{7c}$$

$$C_2 = \mathbf{d}^{\mathrm{T}} \mathbf{c}_0 - \mathbf{r}_j^{\mathrm{T}} \mathbf{d} + \mathbf{r}_j^{\mathrm{T}} \mathbf{r}_j / 2 \tag{7d}$$

Equations (6*a*) and (7*a*) yield

$$c_j = \frac{B_1 C_2 - C_1 B_2}{A_1 B_2 - A_2 B_1}; \quad s_j = -\frac{A_1 C_2 - A_2 C_1}{A_1 B_2 - A_2 B_1} \tag{8}$$

Finally, substituting the above expressions for c_j and s_j into the identity $s_j^2 + c_j^2 = 1$ yields

$$A_1^2 C_2^2 - 2 A_1 C_2 A_2 C_1 + A_2^2 C_1^2 + B_1 C_2 A_1 B_2 - B_1^2 C_2 A_2 - C_1 B_2^2 A_1 + C_1 B_2 A_2 B_1 - A_1^2 B_2^2 + 2 A_1 B_2 A_2 B_1 - A_2^2 B_1^2 = 0 \tag{9}$$

which is the equation for the four-bar linkage *ABCD* applicable to pose *j*. In the equation, the coefficients $A_i, B_i, C_i, i = 1, 2$ are configuration-dependent, but free of ϕ_j. It is seen that the equations are dependent only on the displacements, but not on the orientations. In this light, it is obvious that the problem for dyad synthesis is converted into one of the path generations, rather than one of the motion generations. On the other hand, the synthesis of dyad *EF* is still a problem of rigid-body motion guidance.

4 Extension to Stephenson I and II Linkages

The same approach is further applied to the other two types of Stephenson linkages, as described presently.

4.1 Stephenson-II Linkage. The Stephenson-II mechanism, as shown in Fig. 4, can be constructed by two dyads *AB* and *EF*, together with a third dyad *CD* which connects links *AB* and *EG*. For dyads *AB* and *EF*, the Burmester equation can be directly applied, which leads to

$$\mathbf{b}^{\mathrm{T}} (\mathbf{1} - \mathbf{Q}_j) \mathbf{a}_0 + \mathbf{r}_j^{\mathrm{T}} \mathbf{Q}_j \mathbf{a}_0 - \mathbf{r}_j^{\mathrm{T}} \mathbf{b} + \frac{\mathbf{r}_j^{\mathrm{T}} \mathbf{r}_j}{2} = 0, \quad j = 1, \ldots, m \tag{10}$$

$$\mathbf{f}^{\mathrm{T}} (\mathbf{1} - \mathbf{R}_j) \mathbf{e}_0 + \mathbf{r}_j^{\mathrm{T}} \mathbf{R}_j \mathbf{e}_0 - \mathbf{r}_j^{\mathrm{T}} \mathbf{f} + \frac{1}{2} \mathbf{r}_j^{\mathrm{T}} \mathbf{r}_j = 0, \quad j = 1, \ldots, m \tag{11}$$

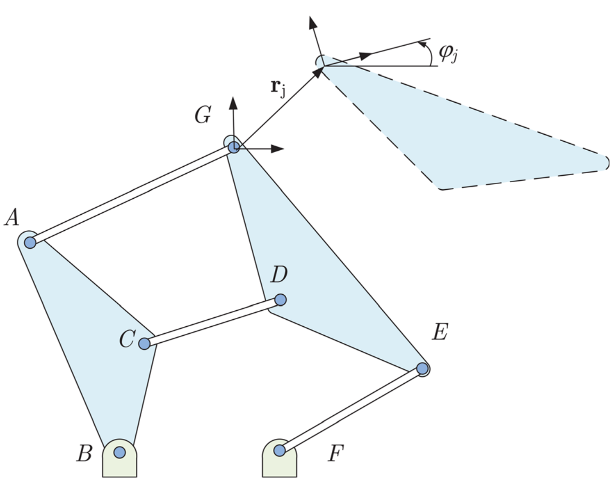


Fig. 4 The Stephenson-II linkage, where dyads *AB*, *CD*, and *EF* are to be synthesized

where $\mathbf{Q}_j$ and $\mathbf{R}_j$ are the rotation matrices for links *AG* and *EG*, respectively.

Link *CD* provides additional kinematic constraints to the kinematic chain *BAGEF* to yield 1DOF mechanism. For link *CD*, the equation below applies

$$||(\mathbf{c}_j - \mathbf{d}_j||^2 = ||\mathbf{c}_0 - \mathbf{d}_0||^2, \quad \text{for } j = 1, \ldots, m \tag{12}$$

where $\mathbf{d}_j = \mathbf{r}_j + \mathbf{R}_j \mathbf{d}_0$.

The coordinate vector of point *C*, $\mathbf{c}_j$, at the *j*th pose, could not be expressed simply in the same way as point *D*, as the orientation of link *AB* is unknown. However, since point *C* is on the same rigid body as *AB*, its coordinates can be parameterized in terms of those of *A* and *B*, as described in the section below.

4.2 Parameterized Coordinates. The parameterized coordinates are applicable to ternary links, for which a point on the link can be obtained by linear combination of any other two points. We take the ternary link *ABC* to describe the parameterized coordinates. Assuming the coordinates of points *A* and *B* are known, the position vector *C* can be expressed as

$$\mathbf{c}_j = \mathbf{b} + \alpha \mathbf{u}_j + \beta \mathbf{v}_j \tag{13}$$

where $\mathbf{u}_j$ and $\mathbf{v}_j$ are orthogonal vectors attached to link *ABC*, which compose a basis of $\mathbb{R}^2$ space. Both vectors are not necessarily unit vectors. Moreover, α and β are dimensionless parameters. Let

$$\mathbf{u}_j = \mathbf{a}_j - \mathbf{b}; \quad \mathbf{v}_j = \mathbf{E}(\mathbf{a}_j - \mathbf{b}) \tag{14}$$

Equation (13) becomes

$$\mathbf{c}_j = \mathbf{b} + \alpha(\mathbf{a}_j - \mathbf{b}) + \beta \mathbf{E}(\mathbf{a}_j - \mathbf{b}) \tag{15}$$

that is

$$\mathbf{c}_j = (\alpha \mathbf{1} + \beta \mathbf{E}) \mathbf{a}_j + [(1 - \alpha) \mathbf{1} - \beta \mathbf{E}] \mathbf{b} \tag{16}$$

which is expressed in a compact form as

$$\mathbf{c}_j = \mathbf{M} \mathbf{a}_j + \mathbf{N} \mathbf{b} \tag{17}$$

with

$$\mathbf{M} = \alpha \mathbf{1} + \beta \mathbf{E}; \quad \mathbf{N} = (1 - \alpha) \mathbf{1} - \beta \mathbf{E} \tag{18}$$

Obviously

$$\mathbf{N} = \mathbf{1} - \mathbf{M} \tag{19}$$

Note that $\mathbf{a}_j = \mathbf{r}_j + \mathbf{Q}_j \mathbf{a}_0$, and hence

$$\mathbf{c}_j = \mathbf{M}(\mathbf{r}_j + \mathbf{Q}_j \mathbf{a}_0) + \mathbf{N} \mathbf{b} \tag{20}$$

that is

$$\mathbf{c}_j = \mathbf{M} \mathbf{Q}_j \mathbf{a}_0 + \mathbf{M} \mathbf{r}_j + \mathbf{N} \mathbf{b} \tag{21}$$

In this way, the position vector *C* is expressed as a linear combination of the position vectors of two other points on the same body. The two points are a circlarpoint and a centerpoint, respectively. The parameterized position vectors simplify significantly the equation of constraints.

In the formulations of synthesis by loop-closure equations, the coordinates are expressed as a function of the rotation angle of a driving link. As the rotation angle is configuration-dependent, such an approach leads to as many intermediate (additional) variables as the number of prescribed configurations. Taking the five-pose synthesis as an example, the synthesis equation by loop-closure equations requires using five rotation angles of the crank as intermediate variables. As the newly introduced formulation of parameterized coordinates needs only two parameters, namely, α, β, the new formulation is thus more concise.

It can be shown that the matrices $\mathbf{M}$ and $\mathbf{N}$ are isotropic, as they obey

$$\mathbf{M}^{\mathrm{T}}\mathbf{M} = (\alpha^2 + \beta^2)\mathbf{1}; \quad \mathbf{N}^{\mathrm{T}}\mathbf{N} = ((1-\alpha)^2 + \beta^2)\mathbf{1} \tag{22}$$

Moreover

$$\mathbf{M}^{\mathrm{T}}\mathbf{N} = (\alpha - \alpha^2 - \beta^2)\mathbf{1} - \beta\mathbf{E} \tag{23}$$

$$\mathbf{N}^{\mathrm{T}}\mathbf{M} = (\alpha - \alpha^2 - \beta^2)\mathbf{1} + \beta\mathbf{E} \tag{24}$$

4.3 Synthesis Equations for the Stephenson-II Linkage. With the introduction of parameterized position vector of point C, Eq. (12) is now rewritten as

$$||\mathbf{M}\mathbf{Q}_j\mathbf{a}_0 + \mathbf{M}\mathbf{r}_j + \mathbf{N}\mathbf{b} - (\mathbf{r}_j + \mathbf{R}_j\mathbf{d}_0)||^2 = ||\mathbf{M}\mathbf{a}_0 + \mathbf{N}\mathbf{b} - \mathbf{d}_0||^2, \quad \text{for } j = 1, \ldots, m \tag{25}$$

Expanding and simplifying the equation yield

$$(\mathbf{M}\mathbf{Q}_j\mathbf{a}_0)^{\mathrm{T}}[\mathbf{N}(\mathbf{b}-\mathbf{r}_j) - \mathbf{R}_j\mathbf{d}_0] - [\mathbf{N}(\mathbf{b}-\mathbf{r}_j)]^{\mathrm{T}}\mathbf{R}_j\mathbf{d}_0 - (\mathbf{N}\mathbf{b})^{\mathrm{T}}\mathbf{N}\mathbf{r}_j - (\mathbf{M}\mathbf{a}_0)^{\mathrm{T}}\mathbf{N}\mathbf{b} + (\mathbf{M}\mathbf{a}_0)^{\mathrm{T}}\mathbf{d}_0 + (\mathbf{N}\mathbf{b})^{\mathrm{T}}\mathbf{d}_0 + (\mathbf{N}\mathbf{r}_j)^{\mathrm{T}}(\mathbf{N}\mathbf{r}_j)/2 = 0 \tag{26}$$

Note that Eq. (19) is used here in derivation. Substituting $\mathbf{Q}_j = c_j\mathbf{1} + s_j\mathbf{E}$ into Eq. (26) leads to

$$A_2c_j + B_2s_j + C_2 = 0 \tag{27a}$$

with coefficients

$$A_2 = (\mathbf{M}\mathbf{a}_0)^{\mathrm{T}}[\mathbf{N}(\mathbf{b}-\mathbf{r}_j) - \mathbf{R}_j\mathbf{d}_0] \tag{27b}$$

$$B_2 = (\mathbf{M}\mathbf{E}\mathbf{a}_0)^{\mathrm{T}}[\mathbf{N}(\mathbf{b}-\mathbf{r}_j) - \mathbf{R}_j\mathbf{d}_0] \tag{27c}$$

$$C_2 = -[\mathbf{N}(\mathbf{b}-\mathbf{r}_j)]^{\mathrm{T}}\mathbf{R}_j\mathbf{d}_0 - (\mathbf{N}\mathbf{b})^{\mathrm{T}}\mathbf{N}\mathbf{r}_j - (\mathbf{M}\mathbf{a}_0)^{\mathrm{T}}\mathbf{N}\mathbf{b} + (\mathbf{M}\mathbf{a}_0)^{\mathrm{T}}\mathbf{d}_0 + (\mathbf{N}\mathbf{b})^{\mathrm{T}}\mathbf{d}_0 + (\mathbf{N}\mathbf{r}_j)^{\mathrm{T}}(\mathbf{N}\mathbf{r}_j)/2 \tag{27d}$$

which can be further expended as

$$C_2 = -[\mathbf{N}(\mathbf{b}-\mathbf{r}_j)]^{\mathrm{T}}\mathbf{R}_j\mathbf{d}_0 - (\mathbf{M}\mathbf{a}_0)^{\mathrm{T}}\mathbf{N}\mathbf{b} + (\mathbf{M}\mathbf{a}_0)^{\mathrm{T}}\mathbf{d}_0 + (\mathbf{N}\mathbf{b})^{\mathrm{T}}\mathbf{d}_0 + ((1-\alpha)^2 + \beta^2)(-\mathbf{r}_j^{\mathrm{T}}\mathbf{b} + \mathbf{r}_j^{\mathrm{T}}\mathbf{r}_j/2) \tag{28}$$

which has a unified form with that of the Stephenson-III linkage. Equations (4), (6a), and (27a) compose the system of synthesis equations for the Stephenson-II linkage.

4.4 Stephenson-I Linkage. Compared with the Stephenson-II, the Stephenson-I linkage, shown in Fig. 5, is different only at joint D, which is located on the rotational link EF, rather than on the floating link EG. The position vector of point D needs to be parameterized too. This can be done similar to the Stephenson-II linkage. To this end, let

$$\mathbf{d}_j = \mathbf{U}\mathbf{e}_j + \mathbf{V}\mathbf{f} \tag{29}$$

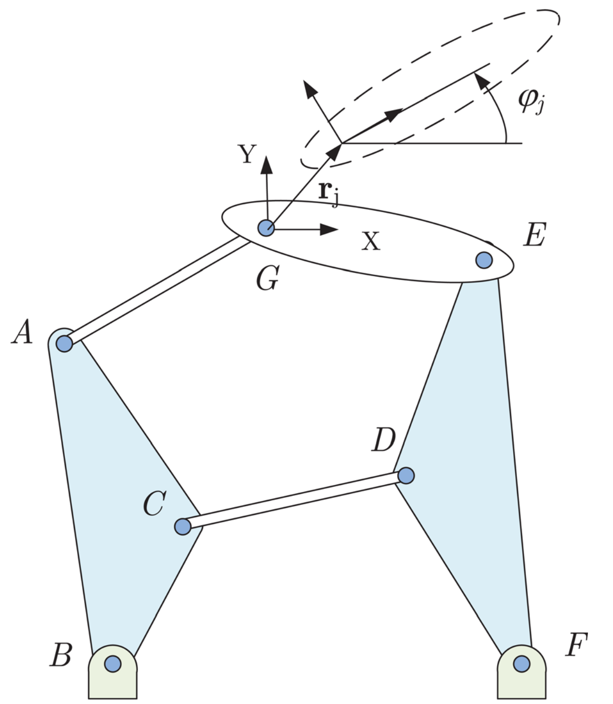


Fig. 5 The Stephenson-I linkage, where dyads *AB*, *CD*, and *EF* are to be synthesized

with

$$\mathbf{U} = \mu\mathbf{1} + \nu\mathbf{E}; \quad \mathbf{V} = (1-\mu)\mathbf{1} - \nu\mathbf{E} \tag{30}$$

where μ and ν are dimensionless parameters.

Substituting Eqs. (17) and (29) into Eq. (12) leads to

$$||\mathbf{M}\mathbf{Q}_j\mathbf{a}_0 + \mathbf{M}\mathbf{r}_j + \mathbf{N}\mathbf{b} - (\mathbf{U}\mathbf{R}_j\mathbf{e}_0 + \mathbf{U}\mathbf{r}_j + \mathbf{V}\mathbf{f})||^2 = ||\mathbf{M}\mathbf{a}_0 + \mathbf{N}\mathbf{b} - \mathbf{U}\mathbf{e}_0 - \mathbf{V}\mathbf{f}||^2 \tag{31}$$

Equation (31), after expanding and simplifying, has the same form as Eq. (27a), with coefficients expressed as

$$A_2 = (\mathbf{M}\mathbf{a}_0)^{\mathrm{T}}(\mathbf{P}\mathbf{r}_j + \mathbf{N}\mathbf{b} - \mathbf{U}\mathbf{R}_j\mathbf{e}_0 - \mathbf{V}\mathbf{f}) \tag{32a}$$

$$B_2 = (\mathbf{M}\mathbf{E}\mathbf{a}_0)^{\mathrm{T}}(\mathbf{P}\mathbf{r}_j + \mathbf{N}\mathbf{b} - \mathbf{U}\mathbf{R}_j\mathbf{e}_0 - \mathbf{V}\mathbf{f}) \tag{32b}$$

$$C_2 = (\mathbf{N}\mathbf{b} - \mathbf{U}\mathbf{R}_j\mathbf{e}_0 - \mathbf{V}\mathbf{f})^{\mathrm{T}}(\mathbf{P}\mathbf{r}_j) - (\mathbf{M}\mathbf{a}_0)^{\mathrm{T}}(\mathbf{N}\mathbf{b} - \mathbf{U}\mathbf{e}_0 - \mathbf{V}\mathbf{f}) - (\mathbf{V}\mathbf{f})^{\mathrm{T}}\mathbf{U}(\mathbf{1} - \mathbf{R}_j)\mathbf{e}_0 + (\mathbf{N}\mathbf{b})^{\mathrm{T}}\mathbf{U}(\mathbf{1} - \mathbf{R}_j)\mathbf{e}_0 + ((\alpha-\mu)^2 + (\beta-\nu)^2)\mathbf{r}_j^{\mathrm{T}}\mathbf{r}_j/2 \tag{32c}$$

where $\mathbf{P} = \mathbf{M} - \mathbf{U}$.

Equations (4), (6a), and (27a) with coefficients A_2, B_2, C_2 defined above compose the system of synthesis equations for the Stephenson-I linkage.

Table 1 Five poses for the example

j	$\mathbf{r}_j$ (mm)	ψ_j
0	$[0, 0]^{\mathrm{T}}$	0
1	$[-92.0, -10.0]^{\mathrm{T}}$	9.2 deg
2	$[-195.0, -35.0]^{\mathrm{T}}$	18.8deg
3	$[-336.0, -106.5]^{\mathrm{T}}$	32.2deg
4	$[-469.4, -344.8]^{\mathrm{T}}$	58.9deg

Table 2 Solutions for the example

Points	Coordinates (mm)	Points	Coordinates (mm)	Points	Coordinates (mm)
B	$[-300, -500]$[a]	A_0	$[-450.0, -200.0]$[a]	D	$[50, -250.0]$
F	$[100.19, -500.08]$	E_0	$[500.08, -200.0]$	$[\alpha, \beta](C)$	$[0.467, -0.40]$

[a]User-specified parameters.

Table 3 Dimensions of a synthesized Stephenson-II linkage (unit: mm)

Link	Length	Link	Length	Link	Length	Link	Length	Link	Length
BA	335.4	BF	400.0	CD	304.1	BC	206.3	AC	223.5
EF	500.0	AG	492.4	EG	538.5	ED	452.7	GD	254.9

So far, we have formulated a unified system of synthesis equations for the three types of Stephenson linkages. With the unified formulation, the three types of Stephenson linkages can be synthesized with the standard formulation as Eqs. (4), (6a), and (7a), where only coefficients A_2, B_2, C_2 need to change accordingly, as summarized in Appendix.

The new formulation offers some advantages over the formulation with loop-closure equations. First, the formulation treats always three dyads, AB, CD, and EF, no matter what type of Stephenson linkage is to be synthesized. On the contrary, the methods with loop closures have to formulate the equations separately for each loop with varying number of links. For example, the Stephenson-III linkage has a four-bar and a five-bar loop, while the Stephenson-II linkage has two five-bar loops.

Second, the new formulation with parameterized coordinates leads to polynomial synthesis equations, while the formulation with loop-closure equations has to deal with trigonometric functions. A system of polynomial equations has normally numeric advantage over trigonometric equations.

5 Synthesis Flexibility of Design With Incomplete Data Sets

Based on the foregoing unified formulation of the synthesis equations, we can observe the following facts:

(a) The dimensional variables in three RR dyads, either in terms of relative coordinates or in terms of parameterized coordinates, amount to 12.
(b) The total number of synthesis equations is equal to $2m$.

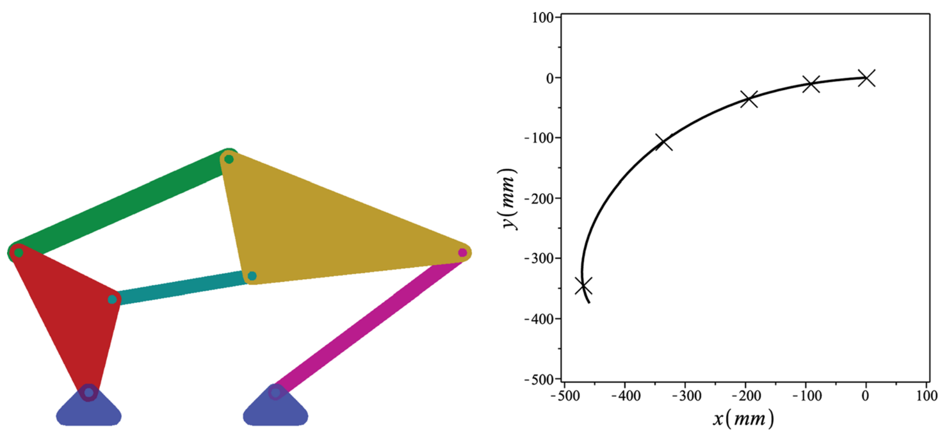
It seems that the system can yield exact solutions for $m = 6$, i.e., seven poses. However, the dyad synthesis equations for dyad EF involve only coordinates of points E_0 and F and admit exact solutions with four equations, i.e., for at most $m = 4$. In other words, the six-bar linkages can admit solutions at most $m = 4$, or, five poses, the same number of poses as the four-bar linkage. On the other hand, there are 12 variables. The six-bar linkage synthesis equations compose an underdetermined system. The problem of six-bar linkage synthesis thus admits infinitely many solutions for five-pose synthesis.

The underdetermined system of synthesis equations implies design flexibility to define a few design parameters by the designer, while exact solutions are still possible. Different selections of design parameters can be considered. Some possibilities include:

(1) Select and define one of the two RR dyads, either AB or CD. In this case, the problem requires only to find the other RR dyad, together with the dyad EF. This is the case of synthesis with constrained 3R chains, as reported by Soh and McCarthy [4].
(2) Select and define two grounding points, i.e., centerpoints, B and D. The problem becomes one to find two circlepoints, A_0 and C_0, together with the dyad EF.
(3) Specify any four coordinates for the four points A_0, B, C_0, and D.

There are other possibilities of selection to be explored [20]. Note that the selection might lead to no real solutions. Thus, how to specify robustly additional conditions (constraints) will be a challenging problem to study.

Due to the fact that the motion synthesis of a six-bar linkage cannot yield a determined system of synthesis equations, but only a underdetermined one, the exact synthesis of a six-bar linkage is essentially a synthesis with incomplete set of pose data. As a matter of fact, the synthesis with an incomplete data set can also be found in other cases, where nonsymmetric kinematic chains exist. For example, the synthesis of spatial revolute-cylindrical-cylindrical-cylindrical (RCCC) linkages involves an incomplete data set too, as the RC and CC dyads require different numbers of prescribed poses for admitting exact solutions. The synthesis with



(a) (b)

Fig. 6 A Stephenson-II linkage synthesized: (*a*) CAD model and (*b*) a piece of coupler curve and points of visit

incomplete pose data stands as a special type of problem and requires due attentions. Given the incompleteness of the data set, additional conditions should be provided to obtain a solution, which implies flexibility in the design to choose freely some design parameters. The selection of parameters with the consideration of robustness is an interesting problem for future study.

6 A Design Example

We provide an example to illustrate the foregoing synthesis method.

The example is given for the synthesis of a Stephenson-II linkage. The set of prescribed data of poses is given in Table 1. As we discussed, the design flexibility of a six-bar linkage allows us to specify pivoting points. In this example, two pivoting points B and D are predefined. With Eqs. (4) and (9), a synthesis result was obtained and listed in Table 2. The synthesis error is equal to 4.9×10^{-6}. The corresponding link dimensions are calculated, as shown in Table 3, for which the linkage is illustrated in Fig. 6. Note that this solution was obtained only for the case with the coordinates of two points B and D predefined. Obviously, there are many feasible solutions can be found, if different pivoting points are specified. This is the so-called design flexibility.

7 Conclusions

The dimensional synthesis of Stephenson linkages was studied. A contribution of the work is the unified formulation of dimensional synthesis equations, which is applicable to the three types of Stephenson linkages. The unified formulation eases the implementation of the dimensional synthesis of six-bar of linkages.

A coordinate parameterizations is proposed, by which the position vector of an R joint on a ternary link is expressed as the linear combination of the position vectors of the other two R joints on the same link. The coordinate parameterization enables the extension of the Burmester theory from the four-bar linkage to the six-bar linkage, and finally leads to the unified formulation of the dimensional synthesis equation of three types of Stephenson linkages. The new method offers advantages over the synthesis methods based on loop-closure equations in the follow aspects:

(1) The introduction of parameterized coordinates avoids using intermediate rotational variables, thus makes the synthesis equation more concise. The number of variables in the synthesis is constant, not varying with the number of poses.
(2) The new method allows the three types of Stephenson linkages to be synthesized with a set of uniform equations, regardless the difference in the kinematic chains of the linkages.
(3) The new formulation yields a system of polynomial equations which can be solved with many available efficient algorithms for polynomial systems.

In this work, the parameterized coordinates are applied to Stephenson linkages. It is possible to extend to other six-bar linkages with ternary links, such as the Watt linkages. Moreover, providing the design flexibility in six-bar linkage motion synthesis, robust selection of design parameters could be addressed in future study.

Acknowledgment

The first author (S.B.) acknowledges the support received from the Haitian Visiting Professorship, Dalian University of Technology, China, and the industrial collaboration with Peder Nielsen Beslagfabrik A/S, Denmark.

Appendix: Parametric Coefficients A_2, B_2, C_2

Stephenson-III Linkage

$$A_2 = \mathbf{r}_j^{\mathrm{T}}\mathbf{c}_0 - \mathbf{d}^{\mathrm{T}}\mathbf{c}_0 \tag{A1a}$$

$$B_2 = \mathbf{r}_j^{\mathrm{T}}\mathbf{E}\mathbf{c}_0 - \mathbf{d}^{\mathrm{T}}\mathbf{E}\mathbf{c}_0 \tag{A1b}$$

$$C_2 = \mathbf{d}^{\mathrm{T}}\mathbf{c}_0 - \mathbf{r}_j^{\mathrm{T}}\mathbf{d} + \mathbf{r}_j^{\mathrm{T}}\mathbf{r}_j/2 \tag{A1c}$$

Stephenson-II Linkage

$$A_2 = (\mathbf{M}\mathbf{a}_0)^{\mathrm{T}}[\mathbf{N}(\mathbf{b} - \mathbf{r}_j) - \mathbf{R}_j\mathbf{d}_0] \tag{A2a}$$

$$B_2 = (\mathbf{M}\mathbf{E}\mathbf{a}_0)^{\mathrm{T}}[\mathbf{N}(\mathbf{b} - \mathbf{r}_j) - \mathbf{R}_j\mathbf{d}_0] \tag{A2b}$$

$$\begin{aligned} C_2 = &- [\mathbf{N}(\mathbf{b} - \mathbf{r}_j)]^{\mathrm{T}}\mathbf{R}_j\mathbf{d}_0 - (\mathbf{M}\mathbf{a}_0)^{\mathrm{T}}\mathbf{N}\mathbf{b} + (\mathbf{M}\mathbf{a}_0)^{\mathrm{T}}\mathbf{d}_0 \\ &+ (\mathbf{N}\mathbf{b})^{\mathrm{T}}\mathbf{d}_0 + \left((1-\alpha)^2 + \beta^2\right)(-\mathbf{r}_j^{\mathrm{T}}\mathbf{b} + \mathbf{r}_j^{\mathrm{T}}\mathbf{r}_j/2) \end{aligned} \tag{A2c}$$

Stephenson-I Linkage

$$A_2 = (\mathbf{M}\mathbf{a}_0)^{\mathrm{T}}(\mathbf{P}\mathbf{r}_j + \mathbf{N}\mathbf{b} - \mathbf{U}\mathbf{R}_j\mathbf{e}_0 - \mathbf{V}\mathbf{f}) \tag{A3a}$$

$$B_2 = (\mathbf{M}\mathbf{E}\mathbf{a}_0)^{\mathrm{T}}(\mathbf{P}\mathbf{r}_j + \mathbf{N}\mathbf{b} - \mathbf{U}\mathbf{R}_j\mathbf{e}_0 - \mathbf{V}\mathbf{f}) \tag{A3b}$$

$$\begin{aligned} C_2 = &(\mathbf{N}\mathbf{b} - \mathbf{U}\mathbf{R}_j\mathbf{e}_0 - \mathbf{V}\mathbf{f})^{\mathrm{T}}(\mathbf{P}\mathbf{r}_j) - (\mathbf{M}\mathbf{a}_0)^{\mathrm{T}}(\mathbf{N}\mathbf{b} - \mathbf{U}\mathbf{e}_0 - \mathbf{V}\mathbf{f}) \\ &- (\mathbf{V}\mathbf{f})^{\mathrm{T}}\mathbf{U}(\mathbf{1} - \mathbf{R}_j)\mathbf{e}_0 + (\mathbf{N}\mathbf{b})^{\mathrm{T}}\mathbf{U}(\mathbf{1} - \mathbf{R}_j)\mathbf{e}_0 \\ &+ \left((\alpha-\mu)^2 + (\beta-\nu)^2\right)\mathbf{r}_j^{\mathrm{T}}\mathbf{r}_j/2 \end{aligned} \tag{A3c}$$

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