The Synthesis of Dyads With One Prismatic Joint

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The classic Burmester problem aims at finding the geometric parameters of a planar four-bar linkage for a prescribed set of finitely separated poses. The synthesis related to the Burmester problem deals with revolute-revolute (RR)*, prismatic-revolute* (PR)*, and revolute-prismatic* (RP) *dyads. A* PR *dyad is a special case of the* RR *dyad, namely, a dyad of this kind with its fixed joint center at infinity; a similar interpretation applies to the* RP *dyad. The special nature of dyads with one* P *joint warrants a special treatment, outside of the general methods of four-bar linkage synthesis, which target mainly* RR *dyads. In proposing robust computational means to synthesize* PR *and* RP *dyads, we adopt an invariant formulation, which, additionally, sheds light on the underlying geometry.* [DOI: 10.1115/1.2829979]

Keywords: Burmester problem, four-bar linkage, parallelism condition, prismatic-revolute dyads

1 Introduction

The Burmester problem aims at finding the geometric parameters of a four-bar linkage for a prescribed set of finitely separated poses.[1] It is well known that RR dyad can be synthesized exactly for up to five prescribed poses. The synthesis problem discussed here pertains to both four and five poses.

The four-pose problem is known to admit infinitely many solutions, each solution dyad being given by a pair of corresponding cubics, the *centerpoint* and the *circlepoint* curves. The five-pose problem, on the other hand, is known to lead to the solution of a quartic equation, and hence, admits none, two, or four real dyads [1]. Extensive research has been reported on the solution of the Burmester problem with different approaches. Bottema and Roth [2], Hunt [3], and McCarthy [4] solved the five-pose problem by intersecting two centerpoint curves of two four-pose problems for two subsets of four poses out of the given five-pose set, to obtain the centerpoints. Beyer [5] and Lichtenheldt [6] reported a method based on projective geometry, while Modler [7–10] investigated various special cases. Sandor and Erdman applied complex numbers [11], while Ravani and Roth [12] and Hayes and Zsombor-Murray [13] solved the problem via the kinematic mapping. Schröcker et al. [14] applied the kinematic mapping to detect the branch defect in the synthesis of four-bar linkages. Recently, Brunnthaler et al. [15] proposed a unified projective-geometric formulation, based on the kinematic mapping, that allows for both dyad-type determination and the dimensional synthesis of the five-pose problem. Furthermore, the Burmester problem has been extended to spatial mechanisms [16]. Except for the kinematic mapping, all foregoing works rely on the location of the poles of the various displacements, which lie at infinity in the presence of a pure translation. To the authors' knowledge, the four-bar linkage synthesis problem with at least one P dyad is not included in commercial design software, such as LINCAGES [17], which provides at most four-pose synthesis [18].

Some new methods for linkage synthesis are proposed recently. Su and McCarthy [19] used polynomial homotopy method to synthesize compliant four-bar mechanisms. Kinzel et al. [20] applied geometric constraint programming to kinematic position synthesis. Kim et al. [21] used spring-connected rigid block models for type synthesis. Hong and Erdman [22] developed an approach to synthesize adjustable four-bar linkages. Perez and McCarthy [23] provided an efficient formulation for the relative kinematics equations by means of the exponential in the Clifford algebra.

The classic Burmester problem, in its full generality, can be stated as follows: A rigid body, attached to the coupler link of a four-bar linkage, as shown in Fig. 1, is to be guided through a discrete set of m poses, given by $\{\mathbf{r}_j, \theta_j\}_0^m$, starting with a reference pose labeled 0, where $\mathbf{r}_j$ is the position vector of a landmark point R of the body at the jth pose, and θ_j is the corresponding angle of a line of the body, as depicted in Fig. 2. The problem consists in finding the joint centers A_0 and B that define the BA_0R dyad of the guiding four-bar linkage, dyad $B^*A_0^*R$ being determined likewise. Given that A_0 and A_0^* describe circles centered at B and B^*, respectively, the former are termed the circlepoints, and the latter the centerpoints of the dyads.

In the balance of this paper, we develop a general synthesis method for four and five poses, applicable to problems admitting either a RR dyad or one dyad with at least one P joint. Such dyads can always be found for the four-pose problem, the conditions for the occurrence of the same dyads in terms of the prescribed set of poses being derived for the five-pose case. A synthesis method, for the same case, is developed by resorting to the geometric conditions for the existence of these dyads. These relations are made apparent by virtue of the frame-invariant formulation adopted in the paper at the outset.

The foregoing conditions, while rather unlikely to occur for an arbitrary set of five poses, should be of interest to the designer, as the set of data poses can always be adjusted in order to "guide" the solution toward a certain type of dyad. We can envision cases in which a five-pose set leads to a RR dyad with its two joint centers a distance orders of magnitude above the lengths of the other three links. In this case we could say that the overly long link is "close" to having a P joint. In this case, it may be more practical to replace this "long" dyad by a PR (or RP) dyad.

2 Synthesis of Linkages With One PR Dyad

We start with the synthesis of the four-bar linkage shown in Fig. 2. Under the usual rigid-body assumption, the *synthesis equation*

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[1]Burmester: "Are there any points in a rigid body whose corresponding positions lie on a circle of the fixed plane for the four arbitrarily prescribed positions?" [1].

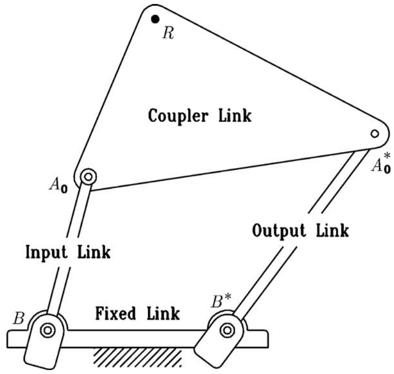

Fig. 1 A four-bar linkage with revolute-revolute dyads

is readily derived:

$$\|\underbrace{(\mathbf{r}_j-\mathbf{b})+\mathbf{Q}_j\mathbf{a}_0}_{\mathbf{a}_j-\mathbf{b}}\|^2=\|\mathbf{a}_0-\mathbf{b}\|^2 \quad \text{for } j=1,\ldots,m \tag{1a}$$

where $\mathbf{a}_0$ and $\mathbf{b}$ are the position vectors of points A_0 and B, the design parameters of the RR dyad, while $\mathbf{Q}_j$ denotes the rotation matrix carrying the guided body from pose 0 to pose j, i.e.,

$$\mathbf{Q}_j=\begin{bmatrix}\cos\phi_j & -\sin\phi_j\\ \sin\phi_j & \cos\phi_j\end{bmatrix} \quad \text{with } \phi_j\equiv\theta_j-\theta_0 \tag{1b}$$

Upon expansion of Eq. (1a) and simplifying the expression thus resulting, we obtain

$$\mathbf{b}^T(\mathbf{1}-\mathbf{Q}_j)\mathbf{a}_0+\mathbf{r}_j^T\mathbf{Q}_j\mathbf{a}_0-\mathbf{r}_j^T\mathbf{b}+\frac{\mathbf{r}_j^T\mathbf{r}_j}{2}=0 \quad j=1,\ldots,m \tag{2}$$

where $\mathbf{1}$ is the 2×2 identity matrix. Equation (2) is the synthesis equations allowing us to compute the design parameters.

To obtain the synthesis equations for PR dyads, we divide both sides of Eq. (2) by the Euclidean norm of $\mathbf{b}$, thus obtaining, for $j=1,\ldots,m$,

$$[(\mathbf{1}-\mathbf{Q}_j)\mathbf{a}_0-\mathbf{r}_j]^T\frac{\mathbf{b}}{\|\mathbf{b}\|}+\left(\mathbf{r}_j^T\mathbf{Q}_j\mathbf{a}_0+\frac{\mathbf{r}_j^T\mathbf{r}_j}{2}\right)\frac{1}{\|\mathbf{b}\|}=0 \tag{3}$$

Furthermore, we define a unit vector β as

$$\beta=\frac{\mathbf{b}}{\|\mathbf{b}\|} \tag{4}$$

When $\|\mathbf{b}\|\to\infty$, the centerpoint B goes to infinity, which leads to a PR dyad, the unit vector β giving the direction of the asymptote of every centerpoint curve arising from every triplet of Eq. (3). Under this condition, the same equation leads to

$$\mathbf{u}_j^T\beta=0 \quad \mathbf{u}_j=\mathbf{r}_j-(\mathbf{1}-\mathbf{Q}_j)\mathbf{a}_0 \quad j=1,\ldots,m \tag{5}$$

where $\mathbf{u}_j\equiv\mathbf{a}_j-\mathbf{a}_0$ is the displacement of the circlepoint A_0 at the jth pose.

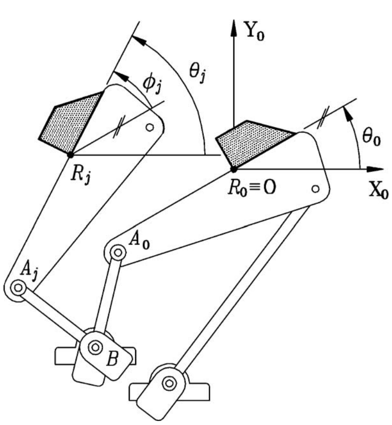

Fig. 2 Two finitely separated poses of a rigid body carried by the coupler link of a four-bar linkage

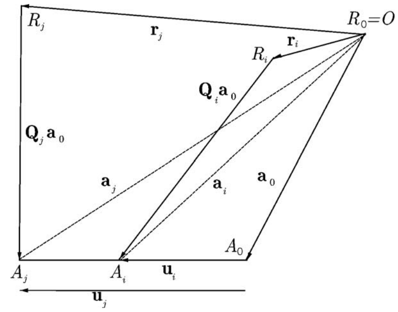

Fig. 3 Relation between the *i*th and *j*th poses and the circlepoints

With reference to Fig. 3, $\mathbf{u}_j$ $(j=1,\ldots,m)$ is the jth displacement vector of the circlepoint. For a PR dyad, all m vectors $\mathbf{u}_j$ must be parallel. In other words, the cross product of any two displacement vectors must vanish. However, rather than working with cross products, we simplify the analysis by resorting to the two-dimensional representation of the cross product introduced in Ref. [24]. This is based on matrix $\mathbf{E}$ rotating vectors in the plane through 90 deg ccw. Hence, the parallelism condition between $\mathbf{u}_i$ and $\mathbf{u}_j$ can be expressed as

$$\Delta_{ij}=\mathbf{u}_j^T\mathbf{E}\mathbf{u}_i=0 \quad i,j=1,\ldots,m \quad i\neq j \tag{6}$$

which expands to

$$\begin{aligned}\Delta_{ij}=&\,\mathbf{a}_0^T(-\mathbf{E}\mathbf{Q}_i-\mathbf{Q}_j^T\mathbf{E}+\mathbf{Q}_j^T\mathbf{E}\mathbf{Q}_i)\mathbf{a}_0\\&-(\mathbf{E}\mathbf{r}_i-\mathbf{Q}_j^T\mathbf{E}\mathbf{r}_i-\mathbf{E}\mathbf{r}_j+\mathbf{Q}_i^T\mathbf{E}\mathbf{r}_j)^T\mathbf{a}_0+\mathbf{r}_j^T\mathbf{E}\mathbf{r}_i=0\end{aligned} \tag{7}$$

We develop below all quadratic terms of Eq. (7), those in the first line of this equation, by writing[2] $\mathbf{Q}_i$ in the form $\mathbf{Q}_i=c_i\mathbf{1}+s_i\mathbf{E}$, in which $s_i\equiv\sin\phi_i$ and $c_i\equiv\cos\phi_i$. Hence,

$$-\mathbf{a}_0^T\mathbf{E}\mathbf{Q}_i\mathbf{a}_0=-\mathbf{a}_0^T\mathbf{E}(c_i\mathbf{1}+s_i\mathbf{E})\mathbf{a}_0=-c_i\mathbf{a}_0^T\mathbf{E}\mathbf{a}_0-s_i\mathbf{a}_0^T\mathbf{E}^2\mathbf{a}_0=s_i\|\mathbf{a}_0\|^2 \tag{8a}$$

$$-\mathbf{a}_0^T\mathbf{Q}_j^T\mathbf{E}\mathbf{a}_0=-\mathbf{a}_0^T\mathbf{E}^T\mathbf{Q}_j\mathbf{a}_0=\mathbf{a}_0^T\mathbf{E}\mathbf{Q}_j\mathbf{a}_0=-s_j\|\mathbf{a}_0\|^2 \tag{8b}$$

$$\begin{aligned}\mathbf{a}_0^T\mathbf{Q}_j^T\mathbf{E}\mathbf{Q}_i\mathbf{a}_0&=\mathbf{a}_0^T(c_j\mathbf{1}-s_j\mathbf{E})(c_i\mathbf{E}-s_i\mathbf{1})\mathbf{a}_0\\&=\mathbf{a}_0^T[-(c_js_i-s_jc_i)\mathbf{1}+(c_js_i+s_jc_i)\mathbf{E}]\mathbf{a}_0\\&=(-c_js_i+s_jc_i)\|\mathbf{a}_0\|^2\\&=-\sin(\phi_i-\phi_j)\|\mathbf{a}_0\|^2\end{aligned} \tag{8c}$$

Further, let $\mathbf{v}_{ij}=-\mathbf{E}\mathbf{r}_i+\mathbf{Q}_j^T\mathbf{E}\mathbf{r}_i+\mathbf{E}\mathbf{r}_j-\mathbf{Q}_i^T\mathbf{E}\mathbf{r}_j$, as that appearing in the second line of Eqs. (7), which are now rewritten as

$$\Delta_{ij}=(s_i-s_j-s_{\overline{ij}})\,\|\mathbf{a}_0\|^2+\mathbf{v}_{ij}^T\mathbf{a}_0+\mathbf{r}_j^T\mathbf{E}\mathbf{r}_i=0$$

$$i,j=1,\ldots \quad m,i\neq j \tag{9}$$

and represent the loci of A_0, of position vector $\mathbf{a}_0$, namely, a family of circles, where $s_{\overline{ij}}=\sin(\phi_i-\phi_j)$.

2.1 Four-Pose Case. In this case, we have three displacements $\{\mathbf{u}_i\}_1^3$, and hence, three parallelism conditions, namely, $\mathbf{u}_1\|\mathbf{u}_2$, $\mathbf{u}_2\|\mathbf{u}_3$, and $\mathbf{u}_3\|\mathbf{u}_1$, as given by Eq. (9) for $(i,j)\in\{(1,2),(2,3),(3,1)\}$. In principle, two of these conditions imply

[2]This idea was first put forth by Bottema and Roth [2].

the third. However, if $\mathbf{u}_2$ happens to vanish, then, while the first two conditions still hold, the third does not necessarily do so. To guarantee the parallelism condition, in any event, we use the three equations (9).

Now, the three equations at hand represent, each, a circle in the x-y plane. It is apparent that we can always find a suitable linear combination of two distinct pairs of the three equations (9) that will yield, correspondingly, two lines. Hence, the parallelism condition leads to one circle $\mathcal{C}$ and two lines $\mathcal{L}_1$ and $\mathcal{L}_2$. The geometric interpretation of the problem of finding the point A_0 then allows a straightforward geometric interpretation: the circlepoint sought (a) is the intersection of the two lines and (b) lies on the circle.

Remarks. (1) If the coefficient of $\|\mathbf{a}_0\|^2$ in one of Eq. (9) vanishes, then the resulting equation is already a line. A second line is then obtained by a suitable linear combination of the two circle equations, which will then lead to the general case. (2) If the same coefficient vanishes in two of Eq. (9), then we need not look for any linear combination to obtain the two lines of the general case. (3) If the same coefficient vanishes in three of Eq. (9), then we have three lines that must be concurrent at a common point.

Once $\mathbf{a}_0$ is obtained, solving for β is straightforward[3]

$$\beta = \frac{\mathbf{E}\hat{\mathbf{u}}}{\|\mathbf{E}\hat{\mathbf{u}}\|} \quad \hat{\mathbf{u}} = \frac{1}{m}\sum_{j=1}^{m}\mathbf{u}_j \tag{10}$$

2.2 Five-Pose Case. Drawing from the case $m=3$, we can conclude that a PR dyad is possible in the case at hand if and only if the asymptotes of the four centerpoint curves $\mathcal{K}_i$ are all parallel. Rather than deriving the parallelism condition for the asymptotes, we resort to an alternative approach, based on that introduced in Sec. 2.1.

In this case, we have four displacements $\{\mathbf{u}_i\}_1^4$, and hence, four parallelism conditions, namely, $\mathbf{u}_1\|\mathbf{u}_2$, $\mathbf{u}_2\|\mathbf{u}_3$, $\mathbf{u}_3\|\mathbf{u}_4$ and $\mathbf{u}_4\|\mathbf{u}_1$, as given by Eq. (9) for $(i,j)\in\{(1,2),(2,3),(3,4),(4,1)\}$. As in the case $m=3$, we enforce the four parallelism conditions (9).

Similarly, we can always find a suitable linear combination of the first equation with each of the remaining three equations (9) that will yield, correspondingly, three lines. Hence, the parallelism condition leads to one circle $\mathcal{C}$ and three lines $\mathcal{L}_1$, $\mathcal{L}_2$, and $\mathcal{L}_3$. A geometric interpretation: the circlepoint A_0 sought (a) is the intersection of the three lines and (b) lies on the circle.

A similar discussion on the cases in which the coefficient of $\|\mathbf{a}_0\|^2$ in Eq. (9) vanishes is straightforward and follows the same line of reasoning as the four-pose case.

If there is one common point to the three lines and the circle, this point is then the solution A_0 sought. Otherwise, there is no solution. After $\mathbf{a}_0$ is obtained, β can be found from Eq. (10).

The synthesis of RP dyads parallels that of PR dyads. Space constraints prevent us from elaborating on this case.

2.3 Solution. In the geometric interpretation of the parallelism conditions, we obtain one circle and two or three lines by manipulating the three or four circle equations. However, it is not recommended to numerically solve the equations thus derived, because of the nature of the operations involved. Indeed, these operations require divisions by the leading coefficients of Eq. (9). The leading coefficients are sums and differences of quantities whose absolute values are not greater than unity. The risk of these coefficients attaining unacceptably small absolute values should not be overlooked, divisions by such numbers being known to lead to numerical catastrophes [25]. A numerically robust solution should be based on the original circle equations, as illustrated below.

[3]One single vector $\mathbf{u}_j$ would suffice. We take the mean value here in order to filter out roundoff-error.

Table 1 Five prescribed poses for rigid-body guidance

j	$\mathbf{r}_j$ (mm)	ϕ_j (deg)
0	$[0.0, 0.0]^T$	0.0
1	$[-2.70366628, 0.650818622]^T$	−13.30931971
2	$[-7.30557983, -0.169753064]^T$	−19.93765489
3	$[-12.19926854, -1.845618865]^T$	−17.30994130
4	$[-16.23615298, -4.100359719]^T$	−6.529901368

3 Numerical Examples

We provide three examples to illustrate the foregoing synthesis procedure. The first and second examples provide solutions to a five-pose problem and a four-pose problem, respectively. The third example shows the robustness of our algorithm.

3.1 Example 1. The first example is the synthesis of a four-bar linkage guiding its coupler link through the five poses of Table 1.

The four cubic centerpoint curves with their asymptotes are shown in Fig. 4; we can see that these asymptotes are parallel to each other. The four circle equations (7) are displayed below:

$$\Delta_{12} = -0.0046389964x^2 - .71629323x - 0.0046389964y^2 - .29517702y + 5.2135630 \tag{11a}$$

$$\Delta_{23} = 0.002389271x^2 - 1.8832828x + 0.002389271y^2 - .97913984y + 11.41245274 \tag{11b}$$

$$\Delta_{34} = 0.0032202972x^2 - 3.2698562x + 0.0032202972y^2 - 1.6663437y + 20.05563909 \tag{11c}$$

$$\Delta_{41} = 0.0015609597x^2 - 3.3158753x + 0.0015609597y^2 - 1.4364879y + 21.652795 \tag{11d}$$

their corresponding circles having one common point, as shown in Fig. 5, which indicates that a PR dyad is possible. Furthermore, the coordinates of this common point yield the unique solution $\mathbf{a}_0$. With $\mathbf{a}_0$ known, β is determined by Eq. (10). The results are recorded in Table 2.

For a RR dyad, the values of $\mathbf{a}_0$ and $\mathbf{b}$, which were obtained using the approach proposed in Ref. 26, are displayed in Table 2. These three solutions correspond to the three intersections in Fig. 6. By inspection, all solutions are found valid for the mechanisms. A synthesized mechanism based on Solution 1 is depicted in Fig. 7, where A_{01} and B_1 are the circlepoint and centerpoint of the RR dyad, respectively, while A_{02} is the circlepoint of the PR dyad. Furthermore, the locations of B_1 and A_{02} are also displayed in Figs. 5 and 6, respectively. In addition to Fig. 7 for Solution 1, two other mechanisms for solutions 2 and 3 are depicted in Fig. 8.

3.2 Example 2. In the second example, the synthesis is conducted by taking the first four poses of Table 1. We have three circle equations (9) in this case. The R joint of the PR dyad is obtained from the intersection of these two circles and the corresponding P joint is found from Eq. (10). The resulting PR dyad in this example is the same as that obtained in Example 1, because the centerpoint curve of this case, as shown in Fig. 9, is just one of the four centerpoint curves in Fig. 4. As for the RR dyads, infinitely many solutions for $\mathbf{a}_0$ and $\mathbf{b}$ are available, namely, any point on the centerpoint and circlepoint curves in Fig. 9.

3.3 Example 3. Let us consider the five poses shown in Table 3. Using the algorithm proposed here, we obtain two circlepoints, $[-0.7676, 2.8467]^T$ and $[-0.8498, 1.9847]^T$, as well as the two corresponding centerpoints, $[-0.3713, 3.3417]^T$ and $[-0.4142, 2.5747]^T$. The results reported in Ref. [27] for the same

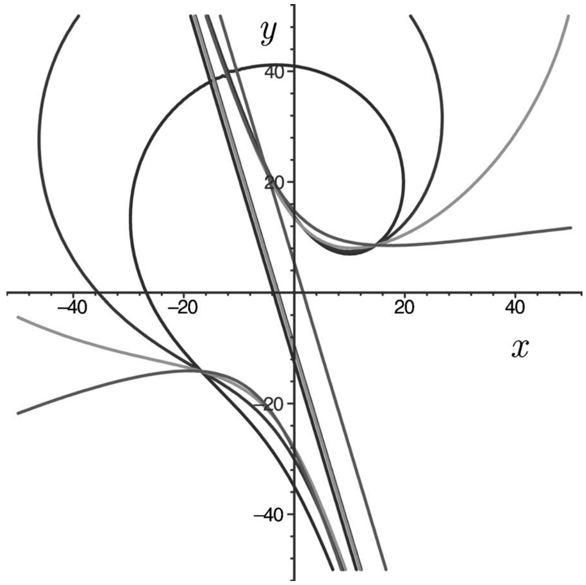

Fig. 4 Four cubic centerpoint curves with their asymptotes parallel to each other

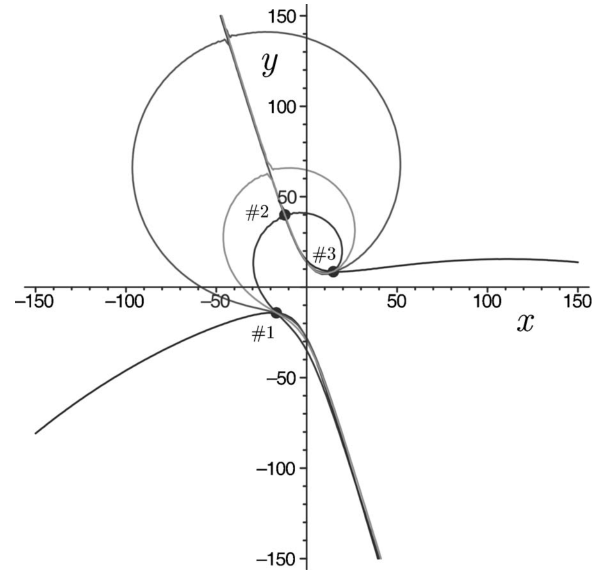

Fig. 6 Contour plots to find the centerpoint of a RR dyad

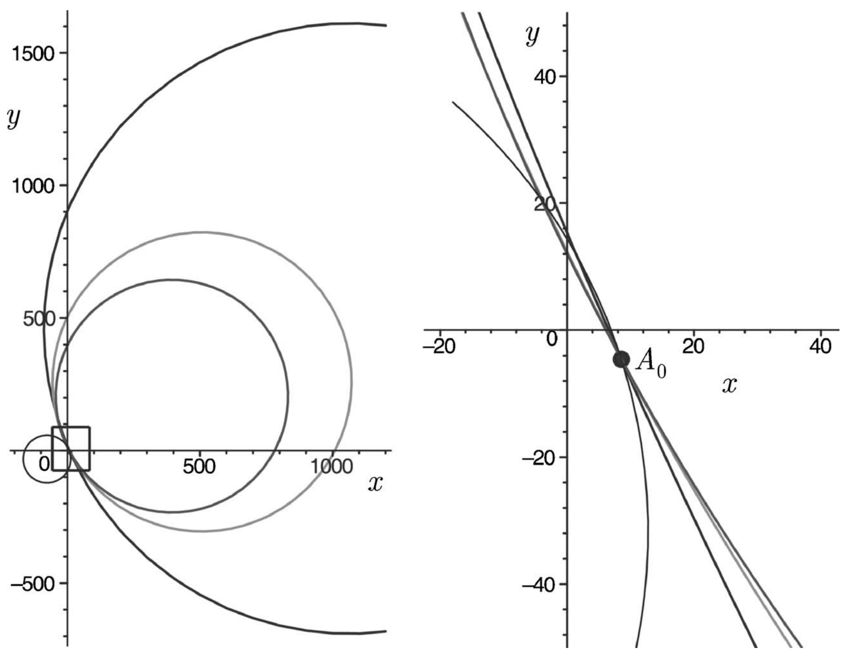

Fig. 5 Contour plots to determine the circlerpoint of a PR dyad: (*a*) the big picture; (*b*) a zoom-in around the common intersection of all circles

Table 2 Synthesis of a four-bar linkage

		$\mathbf{a}_0$ (mm)	β
PR dyad		[8.549330973, −4.559162957]	[−0.2873478855, 0.9578262853]
		$\mathbf{a}_0$ (mm)	$\mathbf{b}$ (mm)
RR dyad	1	[−9.166635761, −7.744210073]	[−16.82708020, −14.17208613]
	2	[7.797896012, 0.882352587]	[−19.33115041, 60.09854120]
	3	[23.74882468, 8.993497071]	[14.72165227, 8.573517789]

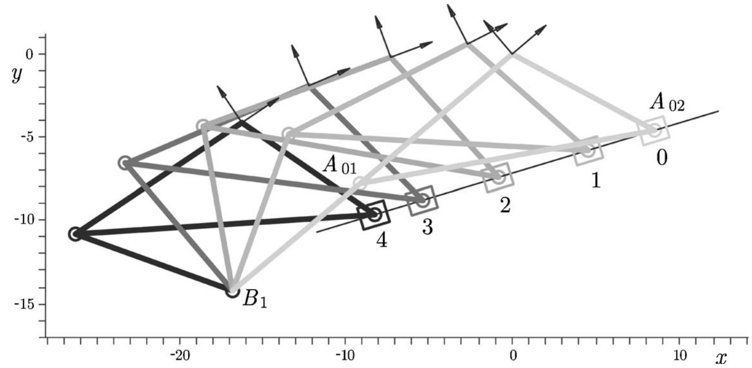

Fig. 7 The linkage synthesized for the five prescribed poses

given poses comprise two circlepoints, $[-0.7600, 2.8370]^T$ and $[-0.9310, 1.9360]^T$, and two centerpoints, $[-0.3640, 3.335]^T$ and $[-0.4840, 2.5150]^T$. The noticeable differences are due to the numerical conditioning of the formulation, rather than to that of the problem itself. The condition number of our solution is about 5, while that occurring with the formulation reported in Ref. [27] is about 150. These results show the advantage of redundancy. That is, by means of a robust algorithm the roundoff error amplification of the data, as that occurring in the results, has been cut down to about 3.33%. To be true, the difference in the two linkages is hardly noticeable, but it becomes apparent when the two are superimposed. The slight difference is the result of a luckily well-conditioned problem.

4 Conclusions

A first contribution of the paper is an alternative approach to the formulation of the Burmester problem. As a second contribution, a method to detect the occurrence of PR and RP dyads, and means to find them in the synthesis of four-bar linkages, is proposed, which obviates the need of asymptote determination. We derived a robust procedure to find the unique orientation of the prismatic joint. We proved that either the circlepoint of a PR dyad or the centerpoint of a RP dyad is located at the common intersection of four circles in the five-pose case. For the four-pose case, a PR or RP dyad always exists. The method introduced here is being in-

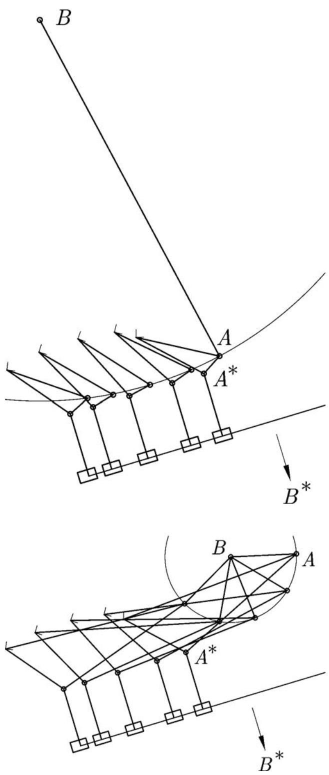

Fig. 8 (*a*) The linkage generated by solution 2 and (*b*) the linkage generated by solution 3

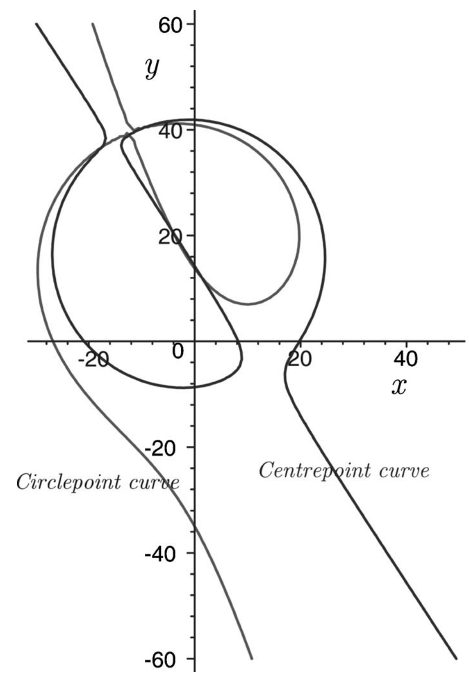

Fig. 9 Contours of center- and circlepoints of the RR dyad

Table 3 Five prescribed poses for rigid-body guidance

j	$\mathbf{r}_j$ (mm)	ϕ_j (deg)
0	$[0.0, 0.0]^T$	0
1	$[1.50, 0.80]^T$	10
2	$[1.60, 1.50]^T$	20
3	$[2.00, 3.00]^T$	60
4	$[2.30, 3.50]^T$	90

tegrated into a web-based design module,[4] which can be used with general-purpose software, for mechanism design.

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