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Coupler-curve synthesis of four-bar linkages via a novel formulation

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ABSTRACT

The coupler-curve synthesis of four-bar linkages is a fundamental problem in kinematics. According to the Roberts–Chebyshev theorem, three cognate linkages can generate the same coupler curve. While the problem of linkage synthesis for coupler-curve generation is determined, it has been regarded as overdetermined, given that the number of coefficients in an algebraic coupler-curve equation exceeds that of linkage parameters available. In this paper, we develop a new formulation of the synthesis problem, whereby the linkage parameters are determined "exactly", within unavoidable roundoff error. A system of coupler-curve coefficient equations is derived, with as many equations as unknowns. The system is thus determined, which leads to exact solutions for the linkage parameters. A method of linkage synthesis from a known coupler-curve equation is further developed to find the three cognate mechanisms predicted by the Roberts–Chebyshev theorem. An example is included to demonstrate the method.

1. Introduction

Path, function and motion generation, classical problems of linkage synthesis, have been extensively studied [1–5]. Given the finite, discrete nature of the linkage parameters, a linkage can be synthesized, in principle, to meet only a finite, discrete set of sampled data from the given path, function or, correspondingly, motion motivating the synthesis problem. Most works on path-generation synthesis have adopted a discretization approach based on Burmester theory. By this approach, a discrete set of points, either specified directly, or taken from a given continuous path, form the synthesis data. Linkage parameters are then found to meet a discrete set of points from the given path. If no more than nine points are specified, exact solutions can be obtained. Synthesis with more than nine points can only be achieved by approximate solutions obtained via optimization methods [6–12].

Quite another approach to path-generation linkage synthesis, rarely considered, is the synthesis of linkages capable of tracing a continuous path. A scarcity of works on this problem has been reported. Blechschmidt and Uicker proposed an approach of synthesis from the algebraic curves of the coupler points [13]. Ananthasuresh and Kota developed a two-step method [14]: (1) curve generation; and (2) parameter evaluation, the latter being conducted approximately. A major challenge in this approach lies in finding the linkage parameters from the coupler-curve equation itself. This is essentially the problem of linkage coupler-curve synthesis, a special but fundamental problem of path synthesis, for which no effective methods are available. The challenge of coupler-curve synthesis lies in the putative overdeterminacy[1] of the synthesis equations. The coupler curve equation of a four-bar linkage is a sextic of 15 terms [14]. As the design parameters are only nine, the synthesis problem appears to be overdetermined, and has been considered as such so far [13,14].

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[1] The overdeterminacy discussed here is confined to the synthesis aimed at generating a full coupler curve, not including synthesis with n discrete points, which is an underdetermined problem if $n < 9$.

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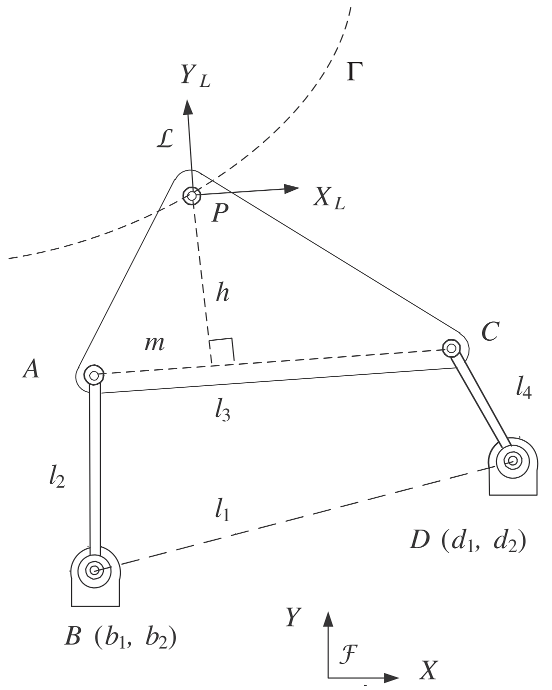


Fig. 1. A four-bar linkage with revolute–revolute (RR) dyads.

In this paper, the coupler-curve synthesis of four-bar linkages is revisited. We propose a new formulation of the synthesis of path generation for four-bar linkages. In doing this, we show that a determined system of coupler-curve coefficient equations can be derived, which allows us to find exact solutions. A related algorithm is developed, then demonstrated with a synthesis example.

2. Problem formulation

A four-bar linkage, illustrated in Fig. 1, is to visit a continuous path Γ traced by a point P of position vector $\mathbf{r}$ in a frame $\mathcal{F}$ fixed to link BD. A local coordinate system $\mathcal{L}$, fixed to the coupler link with origin at point P and its x axis parallel to AC, is also defined. The local coordinates of points A and C are given via their position vectors: $\mathbf{a} = [-m, -h]^T$ and $\mathbf{c} = [l_3 - m, -h]^T$ in the moving frame $\mathcal{L}$. A four-bar linkage is uniquely defined with nine linkage parameters, including: link lengths, $l_2 - l_4$; two coordinates, m and h, defining point P in the coupler link; and the positions of the two joint centers, $B(b_1 b_2)$ and $D(d_1 d_2)$. The length l_1 of the base link is thus calculated from the coordinates of these two points.

We assume that Γ, a coupler curve of the linkage, is given as an algebraic equation in the form

$$f(x, y) = \sum_{i,j=0}^{6} P_{i,j} x^i y^j = 0; \quad i + j \leq 6 \tag{1}$$

where $P_{i,j}$ is the coefficient of term $x^i y^j$, which is known for a given coupler curve. The problem consists in finding the parameters of the linkage tracing Γ. These parameters are the entries of a nine-dimensional array of *design variables*:

$$\mathbf{x} = [m, h, b_1, b_2, d_1, d_2, l_2, l_3, l_4]^T \tag{2}$$

where the link lengths are, of course, non-negative.

In the balance of this paper, the coupler-curve equation is first analyzed, which reveals some apparently elusive properties, upon which a method of linkage synthesis is developed.

3. Coupler-curve equation

The coupler-curve equation (CCE), which is available in the literature, can be readily derived. We include here its derivation for quick reference.[2]

[2] In the authors' opinion, the most elegant derivation of the CCE is given by Bricard, using *isotropic coordinates* [15]. Our derivation here is a bit longer than Bricard's, but more straightforward.

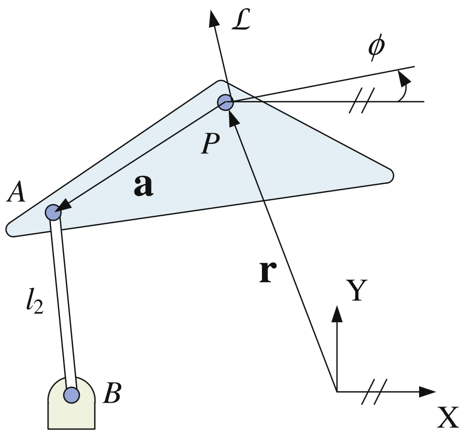


Fig. 2. A RR dyad of a four-bar linkage.

We formulate the CCE based on Burmester theory [16]. Without loss of generality, we start the synthesis with a dyad from a general four-bar linkage, depicted in Fig. 2 at an arbitrary posture. Under the usual rigid-body assumption, the synthesis equation of the dyad *AB* is readily derived for any point of position vector $\mathbf{r}$ on the coupler curve:

$$\|(\mathbf{r}-\mathbf{b})+\mathbf{Q}\mathbf{a}\|^2 = l_2^2 \tag{3}$$

where $\mathbf{b}$ is the position vector of point *B* in the fixed frame $\mathcal{F}$, while $\mathbf{a}$ is that of point *A* of the coupler, expressed in the moving frame $\mathcal{L}$. Moreover, $\mathbf{Q}$ denotes the rotation matrix carrying the coupler link from its reference to its current orientation through an angle ϕ. At the reference position, frame $\mathcal{L}$ takes the same orientation as frame $\mathcal{F}$.

Eq. (3) implies that the trajectory of point *A* is a circle of radius l_2, centered at point *B*. Upon expansion of Eq. (3) and simplifying the expression thus resulting, we obtain

$$(\mathbf{r}-\mathbf{b})^T\mathbf{Q}\mathbf{a}+\frac{1}{2}\left[(\mathbf{r}-\mathbf{b})^T(\mathbf{r}-\mathbf{b})+\mathbf{a}^T\mathbf{a}-l_2^2\right]=0 \tag{4}$$

Similarly, the synthesis equation for the *CD* dyad is obtained as

$$(\mathbf{r}-\mathbf{d})^T\mathbf{Q}\mathbf{c}+\frac{1}{2}\left[(\mathbf{r}-\mathbf{d})^T(\mathbf{r}-\mathbf{d})+\mathbf{c}^T\mathbf{c}-l_4^2\right]=0 \tag{5}$$

where $\mathbf{d}$ is the position vector of point *D* in the fixed frame $\mathcal{F}$, while $\mathbf{c}$ is the position vector of point *C* of the coupler link in the moving frame $\mathcal{L}$.

Angle ϕ is a motion variable, not a linkage parameter. It is thus eliminated as described below.

We expand all terms of Eq. (4) by writing $\mathbf{Q}$ in the form $\mathbf{Q} = c\mathbf{1} + s\mathbf{E}$, in which $\mathbf{1}$ is the 2×2 identity matrix, while $s \equiv \sin\phi$ and $c \equiv \cos\phi$. Furthermore, $\mathbf{E}$ is the 2×2 rotation matrix through 90°. Hence, Eq. (4) can be written as

$$A_1 c + B_1 s + C_1 = 0 \tag{6a}$$

with coefficients

$$A_1 = \mathbf{r}^T\mathbf{a}-\mathbf{b}^T\mathbf{a} \tag{6b}$$

$$B_1 = \mathbf{r}^T\mathbf{E}\mathbf{a}-\mathbf{b}^T\mathbf{E}\mathbf{a} \tag{6c}$$

$$C_1 = \frac{1}{2}\left[(\mathbf{r}-\mathbf{b})^T(\mathbf{r}-\mathbf{b})+\mathbf{a}^T\mathbf{a}-l_2^2\right]. \tag{6d}$$

Likewise, the synthesis Eq. (5) for dyad *CD* leads to

$$A_2 c + B_2 s + C_2 = 0 \tag{6e}$$

with

$$A_2 = \mathbf{r}^T\mathbf{c} - \mathbf{d}^T\mathbf{c} \tag{6f}$$

$$B_2 = \mathbf{r}^T\mathbf{Ec} - \mathbf{d}^T\mathbf{Ec} \tag{6g}$$

$$C_2 = \frac{1}{2}\left[(\mathbf{r}-\mathbf{d})^T(\mathbf{r}-\mathbf{d}) + \mathbf{c}^T\mathbf{c} - l_4^2\right]. \tag{6h}$$

As c and s appear linearly in Eqs. (6a) and (6e), they can be solved for in terms of the coefficients of those equations:

$$c = \frac{B_1C_2 - C_1B_2}{A_1B_2 - A_2B_1}; \quad s = -\frac{A_1C_2 - A_2C_1}{A_1B_2 - A_2B_1}. \tag{6i}$$

Finally, substituting the above expressions into $s^2 + c^2 = 1$ yields

$$\begin{aligned} &{A_1}^2{C_2}^2 - 2\,A_1C_2A_2C_1 + {A_2}^2{C_1}^2 + B_1C_2A_1B_2 - {B_1}^2C_2A_2 \\ &-C_1{B_2}^2A_1 + C_1B_2A_2B_1 - {A_1}^2{B_2}^2 + 2\,A_1B_2A_2B_1 - {A_2}^2{B_1}^2 = 0 \end{aligned} \tag{7}$$

which is the equation of the coupler curve for the four-bar linkage, applicable to any point P on the coupler curve.

3.1. Final form of the CCE

The CCE Eq. (7) is a sixth-order bivariate polynomial, its general form being written as Eq. (1). Some characteristics of the coupler curve have been studied in the literature. From knowledge that the circularity[3] of the curve is three, the CCE can be expressed as

$$\begin{aligned} f(x,y) = {} & K_1\left(x^2+y^2\right)^3 + (K_2x + K_3y)\left(x^2+y^2\right)^2 + \left(K_4x^2 + K_5xy + K_6y^2\right)\left(x^2+y^2\right) \\ & + K_7x^3 + K_8x^2y + K_9xy^2 + K_{10}y^3 + K_{11}x^2 + K_{12}xy + K_{13}y^2 \\ & + K_{14}x + K_{15}y + K_{16} = 0. \end{aligned} \tag{8}$$

This means that the coupler curve of four-bar linkages is a special case of the sextic bivariate polynomial of Eq. (1). Eq. (8) can be derived from Eq. (7), when the latter has a circularity of three, using computer algebra.

The first six coefficients in terms of the link parameters in Fig. 1 are

$$K_1 = -\frac{1}{4}\,l_3^2 \tag{9a}$$

$$K_2 = \frac{1}{2}\left(2b_1l_3^2 + d_1l_3^2 - mb_1l_3 + md_1l_3 + hb_2l_3 - hd_2l_3\right) \tag{9b}$$

$$K_3 = \frac{1}{2}\left(2b_2l_3^2 + d_2l_3^2 - mb_2l_3 + md_2l_3 - hb_1l_3 + hd_1l_3\right) \tag{9c}$$

$$\begin{aligned} K_4 = {} & \frac{1}{2}\left({l_3}^2{l_2}^2 - ml_3{l_2}^2 + l_3m{l_4}^2 + 3\,m{b_1}^2l_3 - 3\,hb_2d_1l_3\right. \\ & -mb_1d_1l_3 - 4\,b_1d_1{l_3}^2 - {b_2}^2{l_3}^2 + m{b_2}^2l_3 + 2\,hd_1d_2l_3 \\ & -3\,{b_1}^2{l_3}^2 + 3\,hb_1d_2l_3 - mb_2d_2l_3 + h^2b_1d_1 - 2\,hb_1b_2l_3 \\ & \left.+h^2b_2d_2 + m^2b_1d_1 - 2\,{d_1}^2l_3m + m^2b_2d_2 + m^2{l_3}^2 - {l_3}^3m + {l_3}^2h^2\right) \\ & -\frac{1}{4}\left({b_1}^2h^2 + {b_1}^2m^2 + {b_2}^2h^2 + {b_2}^2m^2 + {d_2}^2m^2 + {d_2}^2{l_3}^2\right. \\ & \left.+{d_2}^2h^2 + {d_1}^2m^2 + {d_1}^2{l_3}^2 + {d_1}^2h^2\right) \end{aligned} \tag{9d}$$

[3] A planar curve in the x–y plane defined by $F(x,y) = F_n + F_{n-1} + \ldots + F_1 + F_0 = 0$, where each F_i is homogeneous of degree i in (x,y), is circular if and only if any F_n is divisible by $x^2 + y^2$. The circularity of a curve is the highest degree of $(x^2 + y^2)$ that is contained in the curve equation.

$$K_5 = hb_1^2 l_3 - hb_2^2 l_3 - hd_1^2 l_3 + hd_2^2 l_3 + 2\, mb_1 b_2 l_3 - 2\, d_1 m d_2 l_3 - 2\, b_1 b_2 l_3^2 - 2\, b_1 d_2 l_3^2 - 2\, b_2 d_1 l_3^2 \tag{9e}$$

$$\begin{aligned} K_6 = \frac{1}{2}\Big(& l_3^2 l_2^2 - m l_3 l_2^2 + l_3 m l_4^2 + m b_1^2 l_3 - 3\, hb_2 d_1 l_3 \\ & - mb_1 d_1 l_3 - 4\, b_2 d_2 l_3^2 - 3\, b_2^2 l_3^2 + 3\, mb_2^2 l_3 - 2\, hd_1 d_2 l_3 \\ & - b_1^2 l_3^2 + 3\, hb_1 d_2 l_3 - mb_2 d_2 l_3 + h^2 b_1 d_1 + 2\, hb_1 b_2 l_3 \\ & + h^2 b_2 d_2 + m^2 b_1 d_1 - 2\, d_2^2 l_3 m + m^2 b_2 d_2 + m^2 l_3^2 - l_3^3 m + l_3^2 h^2 \Big) \\ - \frac{1}{4}\Big(& b_1^2 h^2 + b_1^2 m^2 + b_2^2 h^2 + b_2^2 m^2 + d_2^2 m^2 + d_2^2 l_3^2 \\ & + d_2^2 h^2 + d_1^2 m^2 + d_1^2 l_3^2 + d_1^2 h^2 \Big). \end{aligned} \tag{9f}$$

The remaining coefficients, extremely lengthy expressions, are not included here; they are available at the first author's website.

The coefficients of Eq. (8) can be expressed in a convenient form. To this end, we rewrite the equation by introducing coefficients $k_i = K_{i+1}/K_1$, $i = 1, \ldots, 15$, as $K_1 \neq 0$, which yields a bivariate polynomial of exactly sixth degree:

$$\begin{aligned} f(x,y) = {} & \left(x^2+y^2\right)^3 + (k_1 x + k_2 y)\left(x^2+y^2\right)^2 + \left(k_3 x^2 + k_4 xy + k_5 y^2\right)\left(x^2+y^2\right) \\ & + k_6 x^3 + k_7 x^2 y + k_8 x y^2 + k_9 y^3 + k_{10} x^2 + k_{11} xy + k_{12} y^2 \\ & + k_{13} x + k_{14} y + k_{15} = 0. \end{aligned} \tag{10}$$

In the above equation, all coefficients are functions of the nine design variables of array x, defined in Eq. (2). In total, we have 15 coefficients $\{k_i\}_1^{15}$, all functions of nine independent variables that are to be found in a synthesis problem. This result tallies with a classical property of planar four-bar linkages: one of its coupler-link points—referred to as *coupler points*—can visit up to nine given points in the plane [17].

4. A determined system of coefficient equations

In coupler-curve synthesis, Eq. (10), comprising 15 coefficients, is assumed to be given. On the other hand, the linkage is defined by only nine parameters. For this reason, the synthesis problem has been considered overdetermined in the literature [14].[4] Below we show that the problem is determined.

We recall that $K_1 = -l_3^2/4$, and introduce a parameter

$$r = l_3/2 \tag{11}$$

which leads to

$$k_i = -K_{i+1}(\mathbf{x})/r^2, \quad i = 1, \ldots, 15. \tag{12}$$

Substituting $l_3 = 2r$ into all coefficients k_i yields new coefficient expressions in terms of l_2, l_4, m, h and the coordinates of the base points $(b_1, b_2)(d_1, d_2)$, along with parameter r, thereby ending up with exactly nine unknowns to be found.

Further analysis on the coefficient functions reveals interesting properties:

(1) Coefficients k_1, k_2 and k_4 are independent of l_2 and l_4
(2) Coefficients k_3 and k_5 are linear in l_2^2 and l_4^2; moreover, the coefficient of l_2^2 in k_3 is identical to its coefficient in k_5. The same holds for the term l_4^2
(3) Coefficient pairs (k_6, k_8) and (k_7, k_9) exhibit the same features as the pair (k_3, k_5)
(4) Coefficients $k_{10,\ldots,}k_{15}$ contain terms including l_2^4, $l_2^2 l_4^2$, l_4^4, l_2^2, l_4^2 only.

These properties allow us to build a system of equations with a reduced number of unknowns, namely, a system of seven equations with seven unknowns, through algebraic manipulations. In this vein, a seven-dimensional array of design variables is defined as

$$\mathbf{y} = [r, m, h, b_1, b_2, d_1, d_2]^T. \tag{13}$$

[4] In fact, the overdeterminacy is not explicitly mentioned, but treated as such.

With any given algebraic equation of a four-bar linkage coupler curve, a system of coefficient equations with all nine design variables can be established:

$$k_i(\mathbf{x}) - k_i^* = 0; \quad i = 1, \ldots, 15 \tag{14}$$

where $\{k_i^*\}_1^{15}$ are the known coefficients of the given coupler curve.

Note that three coefficient functions k_1, k_2 and k_4 are independent of l_2 and l_4. They can be expressed in terms of the seven-dimensional array $\mathbf{y}$, defined above, as

$$k_j(\mathbf{y}) - k_j^* = 0, \quad j = 1, 2, 4 \tag{15a}$$

which yields the first three equations in $\mathbf{y}$ alone.

Variables l_2 and l_4 can be eliminated from the other coefficient equations. This is done by subtracting one from the other in each pair of the coefficient equations of (k_3, k_5), (k_6, k_8) and (k_7, k_9), which leads to three new equations:

$$k_3 - k_5 - (k_3^* - k_5^*) = 0 \tag{15b}$$

$$k_6 - k_8 - (k_6^* - k_8^*) = 0 \tag{15c}$$

$$k_7 - k_9 - (k_7^* - k_9^*) = 0 \tag{15d}$$

thereby obtaining three additional equations in $\mathbf{y}$ alone, for a total of six such equations.

The six remaining coefficients $k_{10}, \ldots, k_{15}$ contain terms in l_2^4, $l_2^2 l_4^2$, l_4^4, l_2^2, l_4^2 only. These equations can be written as:

$$t_{i,1} l_2^4 + t_{i,2} l_2^2 l_4^2 + t_{i,3} l_4^4 + t_{i,4} l_2^2 + t_{i,5} l_4^2 + t_{i,6} = 0; \quad i = 1, \ldots, 6 \tag{15e}$$

where $t_{i,j}$, $j = 1, \ldots, 6$, are parametric coefficients in terms of $\mathbf{y}$, obtained from the $(i + 9)$th equation of (14) for the terms in the above equation. Eq. (15e) can be cast in linear homogenous form, namely,

$$\mathbf{Tq} = 0 \tag{15f}$$

where $\mathbf{q} = [l_2^4, l_2^2 l_4^2, l_4^4, l_2^2, l_4^2, 1]^T$, and $\mathbf{T}$ is a 6×6 matrix whose entries are the ordered array of the coefficients of the components of $\mathbf{q}$, as appearing in Eq. (15e). These entries are functions of $\mathbf{y}$ alone. Since $\mathbf{q} \neq 0$, Eq. (15f) implies

$$g(\mathbf{y}) = \det(\mathbf{T}) = 0 \tag{15g}$$

which completes, with Eqs. (15a)–(15d) and (15g), a system of seven coupler-curve coefficient equations in the seven unknowns of $\mathbf{y}$. Upon solving for $\mathbf{y}$ from the foregoing seven-dimensional system, l_2 and l_4 are the only remaining unknowns, which can be determined utilizing properties (2) and (3) listed above, thereby finding all unknowns.

From the above formulation, we draw an important result:

> *If the coupler curve equation of a four-bar linkage is known, a determined system of coefficient equations can be established, whose solutions yield the complete set of the parameters defining the linkage.*

To the authors' knowledge, this is the first time that the determinacy of the synthesis problem is shown. For decades, researchers treated the problem as overdetermined, even if they were aware of the existence of three cognate linkages for one coupler curve. The problem is, in fact, determined.

4.1. Linkage synthesis

The determined problem formulated above leads to solutions for exact synthesis. A possible method to solve this problem is continuation [18]. However, the high-order multivariate polynomial Eq. (15g), which amounts to 24 with our formulation, implies a big numerical problem for the equation solver. In this light, we propose here an alternative method to solve the synthesis equations, in which a small system of lower Bezout number [19] is solved.

Our method is iterative, with r, defined in Eq. (11), playing a key role in reducing the algebraic complexity, which is measured in terms of the Bezout number at hand. For each given value of r, Eqs. (15a)–(15d) make up a system of six equations in six unknowns, a smaller system with low computational complexity, for which the solutions of the other unknowns in $\mathbf{y}$ are readily found. These solutions are then substituted back into two coefficient equations, for example, k_5 and k_6, to find corresponding solutions for l_2 and l_4. These solutions are then substituted into the remaining equations, while recording the rms error in meeting these equations. The solution with a minimum rms error is chosen as a solution linkage.

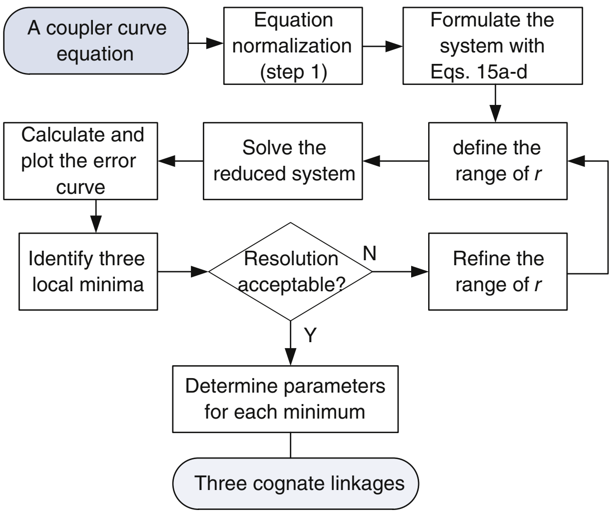


Fig. 3. Flow diagram for linkage parameter determination.

Note that there are multiple real solutions for each r; we thus take the minimum, which means, $e = \min\{s_i\}_1^n$, where $\{s_i\}_1^n$ is the set of all rms values for n real solutions.

Based on this approach, an algorithm is developed for the synthesis problem from a given coupler-curve equation:

1. Start the synthesis with a coupler curve in the form of Eq. (10). The equation needs to be normalized to *monic form*, i.e., to a form in which the coefficient of the term $(x^2 + y^2)^3$ is unity.
2. Identify coefficients k_i^* from the coupler curve equation, as defined in Eq. (14).
3. Define a suitable range of r, then discretize r with a "reasonable" resolution.
4. Calculate residuals with each discrete value of r, following the steps below:

   (a) Solve for m, h, b_1, b_2, d_1, d_2 from Eqs. (15a)–(15d); multiple real solutions are obtained.

   (b) For each solution of Step 4a, find corresponding values of l_2 and l_4 from two coefficient Eq. (14), for example, k_5 and k_6, which are linear in l_2^2 and l_4^2 and lead to unique solutions of both parameters.

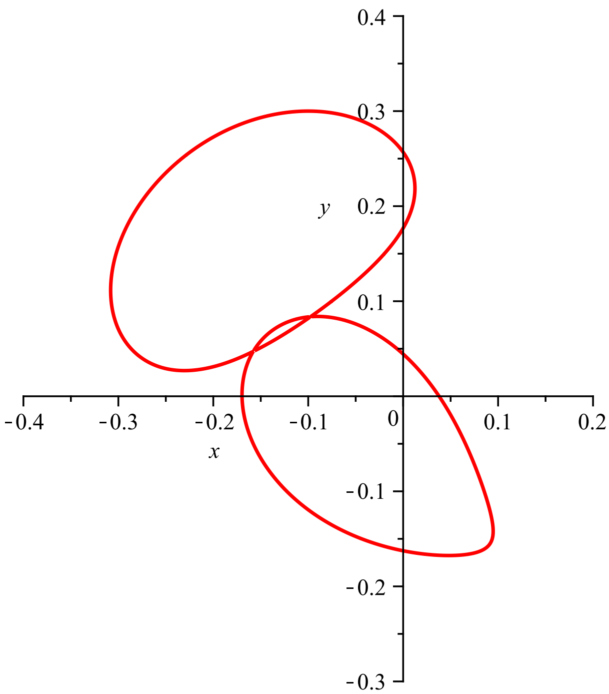


Fig. 4. Coupler curve with two circuits.

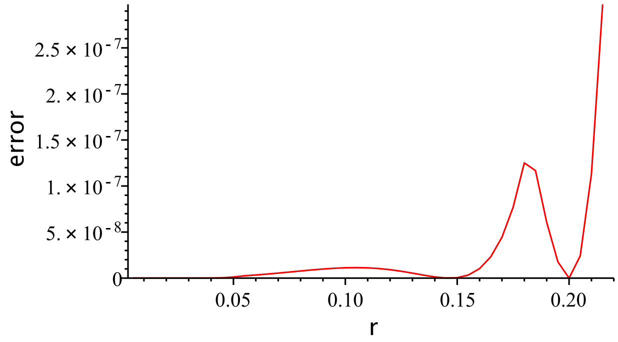


Fig. 5. Plot of error as a function of r.

(c) Calculate the rms error of all six Eq. (15e) for all solutions. The maximum error is recorded, along with the corresponding solutions.

5. Display the error plot and identify visually all minima over the entire plot.
6. Go to Step 3 for a higher resolution around a minimum candidate until this minimum is identified within an acceptable accuracy.
7. Determine all sets of solutions for all minima. Normally, three minima can be expected to yield three sets of solutions, which correspond to the three cognate linkages.

The algorithm is summarized in the diagram shown in Fig. 3.

The proposed method is able to reduce the numerical complexity significantly. The small system of Eqs. (15a)–(15d) contains 2nd, 2nd, 3rd, 3rd, 4th, 4th-order multi-variable polynomials. The Bezout number of the smaller system is equal to 576, much smaller than the larger system including (15 g) with a Bezout number of 13,824 ($=576\times24$). The method can thus readily yield solutions by combining numerical and graphical techniques.

5. Example

We include an example to show the effectiveness of the method. A coupler curve is given via its sextic equation in the form

$$\begin{aligned} f(x,\ y) = {} & x^6+3.0\,x^4y^2+3.0\,x^2y^4+y^6+0.05\,x^5+0.2\,x^4y+0.1\,x^3y^2 \\ & +0.4\,x^2y^3+0.05\,xy^4+0.2\,y^5-0.109375\,x^4+0.18\,x^3y-0.13875\,x^2y^2 \\ & +0.18\,xy^3-0.029375\,y^4+0.00875\,x^3-0.004375\,x^2y-0.01525\,xy^2 \\ & -0.044375\,y^3+0.0107375\,x^2+0.001425\,xy+0.00214375\,y^2 \\ & +0.0008525\,x+0.00107375\,y-0.0000479375025=0 \end{aligned} \tag{16}$$

as displayed in Fig. 4, which has two circuits [20].

Using the foregoing method, the error curve for a range of r values is obtained, as shown in Fig. 5. In the figure, one point, $r=0.20$, appears as a minimizer. Two other minimizers can be found by zooming-in around points $r=0.15$ and $r=0.03$, as shown in Fig. 6. Two local minima, $r=0.034$ and $r=0.147$, are visually obtained. If higher precision is needed, we can always further zoom-in around these points, i.e., to display the e-r plot curve in a short range with a higher resolution.

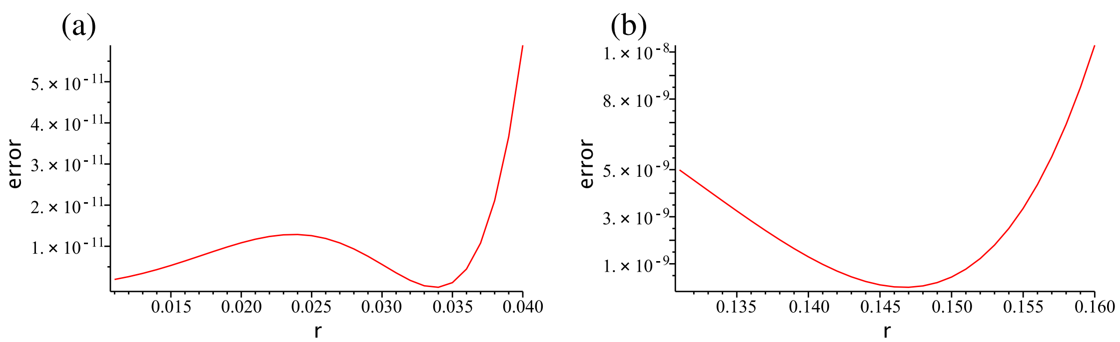


Fig. 6. Zoomed-in views of the error plot around two points, $r=0.03$ and $r=0.15$.

Table 1
Linkage synthesis results for the example (units: m).

No.	r	m	h	$[b_1, b_2]$	$[d_1, d_2]$	l_2	l_3	l_4	rms error
1	0.147	0.3136	0.1568	0.2, − 0.2	− 0.025, 0.1	0.3355	0.294	0.1254	5.40×10^{-7}
2	0.147	− 0.0196	− 0.1568	− 0.025, 0.1	0.2, − 0.2	0.1254	0.294	0.3356	5.40×10^{-7}
3	0.2	0.1	0.15	− 0.2, 0	0.2, − 0.2	0.15	0.4	0.35	0.0
4	0.2	0.3	− 0.15	0.2, − 0.2	− 0.2, 0	0.35	0.4	0.15	0.0
5	0.034	0.083	− 0.125	− 0.2, 0	− 0.025, 0.1	0.180	0.068	0.157	5.71×10^{-8}
6	0.034	− 0.015	0.125	− 0.025, 0.1	− 0.2, 0	0.157	0.068	0.180	5.71×10^{-8}

The method leads to a small system of six equations in six unknowns for any given value of r. As an example, for $r = 0.147$, the equations are

$$hb_2 - hd_2 - mb_1 + md_1 + 0.588\, b_1 + 0.294\, d_1 + 0.00735 = 0 \tag{17a}$$

$$-hb_1 + hd_1 - mb_2 + md_2 + 0.588\, b_2 + 0.294\, d_2 + 0.02940 = 0 \tag{17b}$$

$$\begin{aligned}&2\, hb_1{}^2 - 2\, hb_2{}^2 - 2\, hd_1{}^2 + 2\, hd_2{}^2 + 4\, mb_1 b_2 - 4\, md_1 d_2\\&-1.176\, b_1 b_2 - 1.176\, b_1 d_2 - 1.176\, b_2 d_1 + 0.02646 = 0\end{aligned} \tag{17c}$$

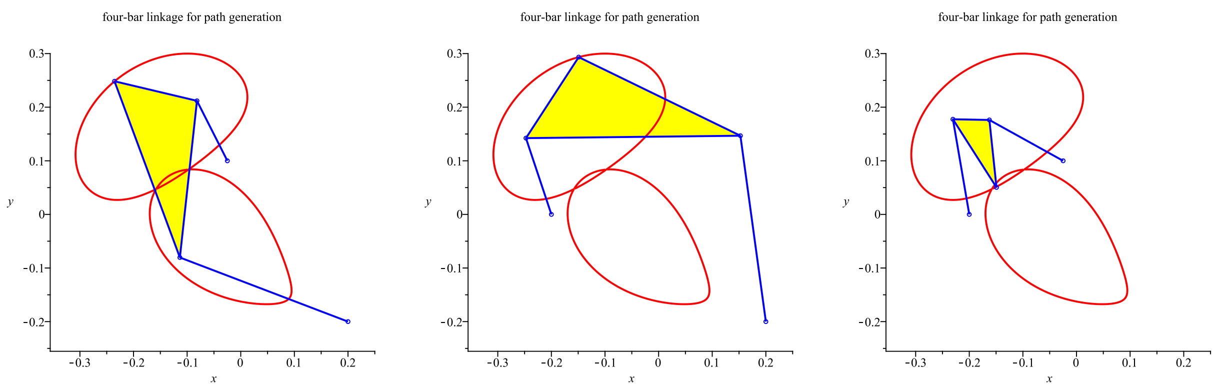
$$\begin{aligned}&-1.176 b_2 d_2 - 2md_2^2 - 2md_1^2 + 2md + 1.176 b_1 d_1 + 2md_1^2\\&-4\, hd_1 d_2 + 4\, hb_1 b_2 + 0.588\, b_1{}^2 - 0.588\, b_2{}^2 + 0.01176 = 0\end{aligned} \tag{17d}$$

$$\begin{aligned}&2.352\, b_1 b_2 d_2 - 1.176\, b_1{}^2 d_1 + 1.176\, b_2{}^2 d_1 + 8\, hb_1 d_1 d_2\\&-8\, hb_1 b_2 d_1 - 8\, mb_1 b_2 d_2 + 8\, mb_2 d_1 d_2 - 4\, hb_1{}^2 d_2 + 4\, hb_2{}^2 d_2 + 4\, mb_1{}^2 d_1\\&-4\, mb_1 d_1{}^2 + 4\, mb_1 d_2{}^2 - 4\, mb_2{}^2 d_1 + 4\, hb_2 d_1{}^2 - 4\, hb_2 d_2{}^2 - 0.003528 = 0\end{aligned} \tag{17e}$$

$$\begin{aligned}&-1.176\, b_1{}^2 d_2 + 1.176\, b_2{}^2 d_2 + 8\, mb_1 b_2 d_1 - 8\, mb_1 d_1 d_2\\&+8\, hb_2 d_1 d_2 - 8\, hb_1 b_2 d_2 - 2.352\, b_1 b_2 d_1 - 4\, mb_2 d_1{}^2\\&+4\, hb_1{}^2 d_1 - 4\, hb_2{}^2 d_1 + 4\, mb_1{}^2 d_2 - 4\, mb_2{}^2 d_2\\&+4\, mb_2 d_2{}^2 - 4\, hb_1 d_1{}^2 + 4\, hb_1 d_2{}^2 - 0.00588 = 0.\end{aligned} \tag{17f}$$

The above system of equations yields six sets of real solutions. By substituting them into the coefficient equations of k_5 and k_6, which are linear in l_2^2 and l_4^2, values of l_2 and l_4 are further obtained. As l_2^2 and l_4^2 have to be positive, in the given example, only two sets of real solutions are obtained for parameters l_2 and l_4. They are the solutions for the given r.

The design parameters are then obtained, as listed in Table 1, along with the minimum rms errors. For $r = 0.2$, the rms error vanishes within the numerical resolution of 20 digits in this work, which shows the validity of the determined system of synthesis equations to yield exact solutions. For each value of r, two sets of solutions with minimum rms error can be found. While the solutions are different, both solutions generate the same linkage. This can be readily explained, as the two dyads BA and DC can be swapped. In total,



(a) (b) (c)

Fig. 7. Three cognate linkages visiting the first circuit of the coupler curve.

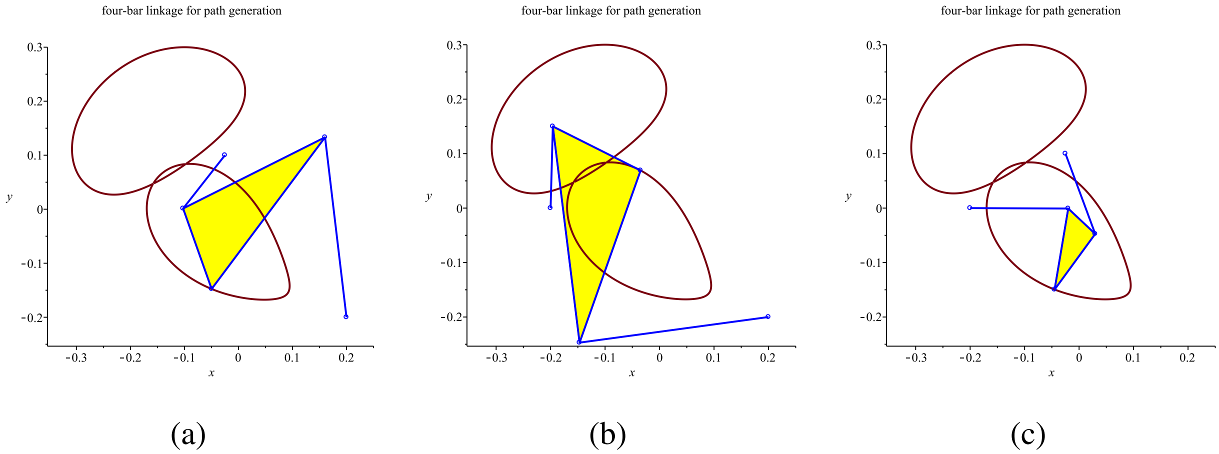


Fig. 8. Three cognate linkages visiting the second circuit of the coupler curve.

we obtain three linkages, which are the cognate linkages anticipated by the Roberts–Chebyshev Theorem [21]. Fig. 7 displays the three linkages, which visit the first circuit of the coupler curve. In Fig. 8, the cognate linkages visiting the second circuit of the coupler curve are displayed.

6. Conclusions

Linkage synthesis for path generation was studied with focus on the coupler curves of four-bar linkages. The problem of synthesis from the algebraic equation of the coupler curve was addressed. We derived a determined system of coefficient equations.

The contributions reported here are summarized below:

- We showed that the system of coefficient equations in the nine linkage parameters is determined. This is achieved based on the analysis of the curve coefficients and proper algebraic manipulation.
- We developed a new method of synthesis in combining a numerical method with graphics tools, which allows us to solve the problem with a smaller system of equations of lower complexity. The method, combining a graphic tool in error displaying, yields directly and simultaneously synthesis solutions of the three cognate linkages. The solution method based on the parameter r makes it possible to solve the synthesis problem graphically by means of error plots; with which the three cognate-linkage solutions are readily identified. To the authors' knowledge, no other method allows for an effective, thorough synthesis.

This work sheds light on a fundamental issue in linkage synthesis: the determinacy of the coupler-curve coefficient equations leads to exact, rather than approximate, solutions for the synthesis of a continuous coupler curve. In this work, an iterative algorithm, with the parameter r as argument, is developed. More elegant and efficient algorithms can be obtained based on our formulation, but these lie outside of the scope of the paper. Moreover, this result can be extended to other types of linkages, including six-bar, spherical and spatial linkages.

The method of synthesis from the coupler curve can be further developed for the synthesis with finitely separated positions, in which a discrete set of $n > 9$ points is prescribed for approximate solutions. To this end, the prescribed points have to fit into a tricircular sextic coupler curve. Once this sextic has been found, our formulation takes over. The challenge to fit the n given points to the most likely coupler curve is an open problem, to be addressed in future studies.

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