The Physics of Compact Objects
Exercises
version 2.33

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1. **The magnetic dipole model**

One can show that the energy loss-rate due to magnetic-dipole radiation is given by:

$$\dot{E}_{dipole} = -\frac{2}{3c^3}|\ddot{\vec{m}}|^2 \quad \text{where} \quad |\ddot{\vec{m}}| = B R^3 \Omega^2 \sin\alpha$$

$\vec{m}$ is the magnetic dipole moment of the pulsar, B is the magnitude of the surface mag. flux density, R is the radius, Ω is the angular frequency (angular velocity, $\Omega = 2\pi/P$ where P is the spin period) and finally α is the angle between the magnetic and rotation axes.

This energy can only be provided by the kinetic rotational energy of the pulsar given by:
$E_{rot} = \frac{1}{2}I\Omega^2$ where I is the moment of inertia (for a uniform solid sphere) $I \simeq \frac{2}{5}MR^2$.

For a "standard" neutron star $M = 1.4 M_\odot$, $R = 10$ km and assume here $\alpha = 90°$.
(throughout these notes use the above values for M, R and I unless stated otherwise).

a) Show that $B = \sqrt{\dfrac{3c^3 I}{8\pi^2 R^6} P\dot{P}} \approx 3\times10^{19} \sqrt{P\dot{P}}$ *Gauss*

b) Use a P-$\dot{P}$ diagram (e.g. Tauris & van den Heuvel, 2023, Fig.1.2) to find typical values for P and $\dot{P}$ and compare the $\vec{B}$-field strength of a typical pulsar with that of a millisecond pulsar.
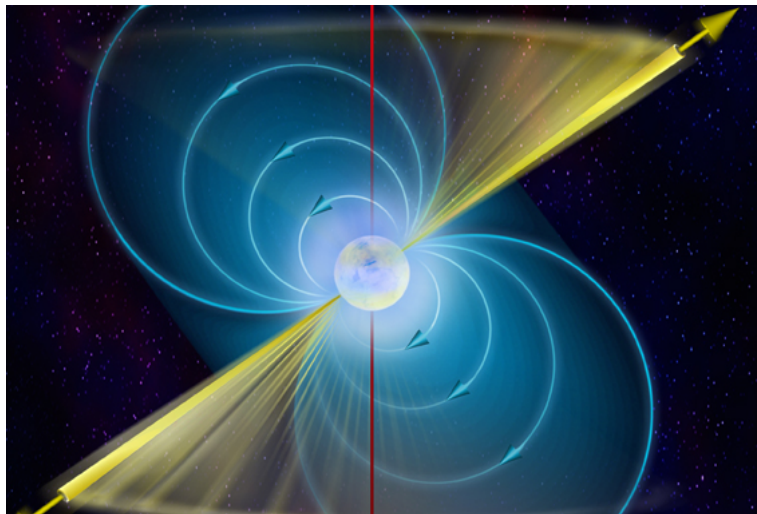
2. Coherent radio emission mechanism

Only a tiny fraction of the energy loss of a spinning radio pulsar is emitted as radio waves from which the pulsars are detected. The energy loss is dominated by electromagnetic dipole radiation with a frequency equal to the rotational frequency of the pulsar.

For the Crab pulsar the radio flux-density of the detected signal at 436 MHz is $\sim 0.48\ Jy$ $\left(1\ Jansky \equiv 10^{-23}\ erg\ s^{-1}\ cm^{-2}\ Hz^{-1}\ St^{-1}\right)$. The distance to the Crab pulsar is 2.0 kpc.

a) Use the P-$\dot{P}$ diagram to estimate P and $\dot{P}$ for the Crab pulsar and show that:

$$\dot{E}_{rot} \approx -5\times10^{38}\ erg\ s^{-1} \qquad \wedge \qquad L_{radio} \approx 10^{31}\ erg\ s^{-1} \qquad \text{i.e. } L_{radio} \leq 10^{-7}\ |\dot{E}_{rot}|$$

Assume (not correct) a flat spectral index, i.e. that the radio emission is equally intense at all frequencies and assume a total frequency bandwidth of 436 MHz.

It should also be noted that for the Crab pulsar a large fraction of the lost rotational energy goes to light-up the Crab nebula via injection of relativistic particles, as well as dissipation caused by torques from magnetospheric currents. Furthermore, a small fraction is needed for the observed magnetospheric emission of optical-, X- and γ-rays.

b) Given the radio luminosity, L_{radio} calculate the surface intensity of the radio emission, I (in units of $erg\ s^{-1}\ cm^{-2}\ Hz^{-1}\ St^{-1}$) and use a Planck function to demonstrate that if the radio emission was caused by thermal black body radiation one would obtain an extremely high brightness temperature (leading to absurdly large particle energies, $E \approx kT$) and therefore the radiation mechanism of a radio pulsar *must* be coherent (most models invoke curvature radiation or a maser mechanism).

(The Planck function is given by $I = \dfrac{2h\upsilon^3}{c^2}\dfrac{1}{e^{h\upsilon/kT}-1}$. Apply the Rayleigh-Jeans law.)
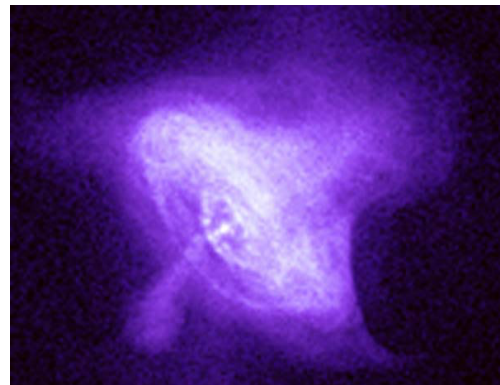
3. Pulsar evolution in the _P-P_{dot}_ diagram

Radio pulsars are born with large magnetic fields ($\sim 10^{13}$ G) due to some unspecified thermo-magnetic effects during, or shortly after, the neutron star formation. Given that the electrical conductivity (σ) in the outer crust of a neutron star is finite, ohmic dissipation causes the B-field to decay over time. Below we will assume that this effect is the only one that affects the dynamical evolution of a radio pulsar (see Tauris & Konar 2001 for further discussion).

The induction is given by: $\dfrac{\partial \vec{B}}{\partial t} = -\dfrac{c^2}{4\pi} \nabla \times \left(\dfrac{1}{\sigma} \times \nabla \times \vec{B} \right) + \nabla \times (\vec{v} \times \vec{B})$

Disregarding convective transport of material (only relevant for accreting neutron stars) we disregard the last term, and thus: $\dfrac{\partial \vec{B}}{\partial t} = -\dfrac{c^2}{4\pi} \nabla \times \left(\dfrac{1}{\sigma} \times \nabla \times \vec{B} \right)$

For the uniform conductivity case, the above equation takes the simple form of a pure diffusion equation (by virtue of the divergence-free condition for the magnetic fields):

$$\dfrac{\partial \vec{B}}{\partial t} = -\dfrac{c^2}{4\pi\sigma} \nabla^2 \vec{B}$$

a) Show that the solution is an exponential decay of the surface B-field: $\quad B(t) = B_0\, e^{-t/\tau_D}$

and write an expression for the decay-time constant, τ_D. Verify the unit for Gauss in cgs.

b) Insert this expression into the B-field obtained from the dipole model:

$B = \sqrt{\dfrac{3c^3 I}{8\pi^2 R^6} P \dot{P}} \;=\; k\sqrt{P\dot{P}} \quad$ and integrate in order to obtain the following expression:

$P(t)^2 = P_0^2 + B_0^2\, \tau_D \left(1 - e^{-2t/\tau_D} \right) \cdot \dfrac{1}{k^2}$

c) Find $\lim\limits_{t \to \infty} P(t)$ for a pulsar born with $B_0 = 10^{13}$ G and initial spin period $P_0 = 20$ ms

assuming a decay constant of $\tau_D \sim 10\,\mathrm{Myr}$ and using $R = 10\,km$ and $I = 10^{45}\, g\,cm^2$.

d) Plot the above evolutionary track in a $P - \dot{P}$ diagram.

4. <u>On the true age of radio pulsars</u>

The characteristic age of radio pulsars is defined by: $\tau \equiv \dfrac{P}{2\dot{P}}$

If the surface B-field of a neutron star decays on a relatively short timescale the characteristic age of a pulsar, τ is an overestimate of it true age, t.

a) Isolate $\dot{P}$ from the equation for the dipole B-field strength: $B = k\sqrt{P\dot{P}} = B_0\, e^{-t/\tau_D}$ and insert it into the expression for the characteristic age, τ. Then insert $P(t)$ from question 3b while noticing that $P_0 \ll P$ and derive: $\tau \simeq \dfrac{\tau_D}{2}\left(e^{2t/\tau_D} - 1\right)$

Hence you have shown that the true age of a pulsar is given by:

$$t \simeq \frac{\tau_D}{2}\ln\left(\frac{2\tau}{\tau_D} + 1\right)$$

b) What is the true age of a radio pulsar if $\tau_D = 10\ Myr$ and it has a characteristic age of $100\ Myr$?

5. <u>Constraints on the neutron star (nuclear) equation of state</u>

Recently a group of observers (Philip Kaaret et al. 2007) have reported the discovery of an extremely fast rotating X-ray pulsar (*XTE J1739-285*) during a thermonuclear burst with a spin frequency of $f \approx 1122\,\text{Hz}$. Here we will set simple constraints on the equation of state as a consequence of this new discovery (Note, this measurement has not been reconfirmed). Consider a nucleon at the surface of a spinning neutron star. Disregard any electromagnetic forces acting on it. A stability criterion must be that the magnitude of the centrifugal force must be smaller than the magnitude of the gravitational force acting on the nucleon in order not to dissolve the fast spinning neutron star, i.e.: $F_{centrifugal} < F_{gravitational}$

a) Assume a neutron star has a spherical shape (radius R) and show that the criterion above leads to the following Newtonian constraint on the neutron star mean mass density:

$$\bar{\rho} > \frac{3\pi}{GP^2}$$

b) Calculate the minimum value of $\bar{\rho}$ (in units of both g/cm³ and $\rho_{nuclear} = 2.8 \times 10^{14}$ g/cm³)

c) i) Considering a neutron star with a radius of $R = 10\ km$, calculate the equatorial rotational speed at the surface of such a pulsar.

ii) Constrain the max. radius of the pulsar *XTE J1739-285* only using $f \approx 1122\,\text{Hz}$.

Notice, this discovery will rule out many so-called *stiff* equations of state since such a high mass density can only be achieved with a *soft* equation of state (or exotic strange stars).

d) Estimate the length of a queue of cars whose total mass equals that of 1 cm³ of material from the centre of a neutron star, where the density is assumed to be $\sim 5\,\rho_{nuclear}$

($\rho_{nuclear} \approx 2.8 \times 10^{14}\ g\ cm^{-3}$)
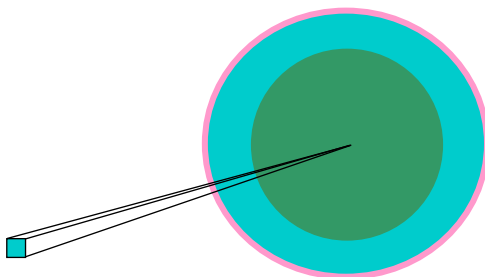
6. <u>Gravitational redshift of a photon emitted from a neutron star surface</u>

A photon travelling out through the gravitational field of a compact object will lose energy.
An estimate of the resulting change in photon frequency is given by:

$$\frac{df}{f} = -g\frac{dh}{c^2}$$ where $dh = dr = r_2 - r_1$ is the radial distance covered, and $g = G\frac{M}{r^2}$ is the

corresponding local gravitational field strength at distance r from mass M (see fig. below).

a) Consider a violet photon ($\lambda = 400$ nm) emitted from the surface of a neutron star.
   What will be the color (wavelength) of the photon when detected at the Earth ?

(Hint: neglect the gravitational field of the Earth. Integrate and show: $f_2 = f_1 e^{+\Phi(r_1)/c^2}$,

where $\Phi(r)$ is the gravitational potential at distance r).

b) In text books one finds the weak-field limit of the gravitational redshift as: $\dfrac{\Delta\lambda}{\lambda} = \dfrac{GM}{Rc^2}$

Derive this result from $\dfrac{\Delta\lambda}{\lambda} \simeq \dfrac{\sqrt{1-2GM_{Earth}/R_{Earth}c^2}}{\sqrt{1-2GM/Rc^2}} - 1$ considering a weak field: $\dfrac{GM}{Rc^2} \ll 1$

and neglecting the gravitational field of the Earth (leading to negligible photon blueshift).
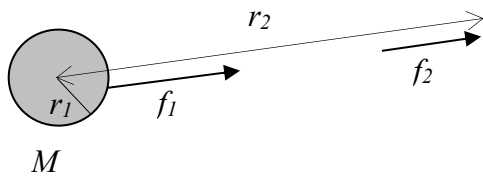
Schwarzchild metric, exterior geometry: $\qquad e^{2\Phi/c^2} = 1 - \dfrac{2GM}{Rc^2}$

7. <u>Gravitational wave radiation (gwr) from a 1 mm "mountain" on a neutron star surface</u>
If a rotating body is asymmetric with respect to its rotation axis there will be emission of
gravitational waves. The luminosity of this radiation depends on the third time-derivative of
the quadrupole moment of the spinning body, which depends on its mass distribution, size
and spin rate. Consider a neutron star with radius, R and a mountain of height, Δr located on
the equator. One can show that the gravitational wave energy-emission rate is given by:

$$L_{gwr} \equiv \frac{|dE|}{dt} = \frac{32\,G}{5\,c^5} \cdot I^2 \varepsilon^2 \Omega^6 \quad \text{where we now assume}$$

$\varepsilon \approx \dfrac{\Delta r}{R}$ is the ellipticity (here: ratio of mountain height to neutron star radius)

$I \approx \frac{2}{5} MR^2$ is the moment of inertia, and $\Omega = 2\pi / P$ is the angular velocity

Calculate the energy loss-rate due to emission of gravitational waves caused by a 1 mm
mountain on the neutron star equator with a spin period, $P = 5$ ms and compare with the
rotational energy-loss rate of a typical millisecond pulsar. What is your conclusion ?
(use $R = 10$ km., $M = 1.4\,M_\odot$ and find a value for $\dot{P}$ in the $P - \dot{P}$ diagram).

8. <u>Merger time-scale of compact binaries due to emission of gravitational waves</u>

The relative rate of change of orbital angular momentum in
a circular binary emitting gravitational waves (GWs) is given by:

$$\frac{\dot{J}_{gwr}}{J} = -\frac{32}{5}\frac{G^3}{c^5}\frac{M_1 M_2 (M_1 + M_2)}{a^4}$$
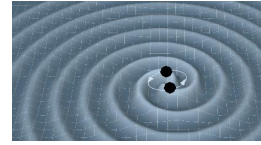
Hence it follows that: $\quad \frac{1}{J}\,dJ = k_1 \frac{1}{a^4}\,dt \quad$ where $\quad k_1 = -\frac{32}{5}\frac{G^3}{c^5} M_1 M_2 (M_1 + M_2)$

Given the definition of orbital angular momentum and Kepler's 3. law one has:

$$J \equiv |\vec{r} \times \vec{p}| = \mu a^2 \Omega = \frac{M_1 M_2}{M_1 + M_2} a^2 \Omega \qquad \wedge \qquad \Omega^2 = \frac{G(M_1 + M_2)}{a^3}$$

a) Combine the above equations and show that: $\dfrac{1}{J}\,dJ = k_1\,k_2^4\,\dfrac{1}{J^8}\,dt \;$ where $\; k_2 = G\dfrac{M_1^2 M_2^2}{M_1 + M_2}$

b) Integrate this equation and show that the merger time-scale, τ is given by: $\quad \tau = \dfrac{1}{8}\left|\dfrac{J}{\dot{J}}\right|$

c) Measuring orbital separation, a in $R_\odot$ and masses in $M_\odot$ derive:

$$\tau = 150\,\frac{a^4}{M_1 M_2 (M_1 + M_2)}\;\; Myr$$

d) The double pulsar binary PSR J0737-3039 ($M_1 \approx M_2 \approx 1.3 M_\odot$) discovered in 2003 has
an orbital period of 2.4 *hours* and an eccentricity of ~0.1. Assume the binary is circular
and find a via Kepler's 3. law and show the merger time-scale is only about 80 *Myr*.

Although the system merges in about 80 Myr, and it is relatively nearby at a distance of
~ 1.15 kpc, its continuous low-frequency GW emission will hardly be detectable by LISA.

Note: in eccentric binaries the merger time-scale becomes substantially shorter (Peters 1963)

9. **Direct wind mass loss**
   In this problem we study the effects of mass loss on the orbital evolution of a binary system. Consider two stars with masses M_1 and M_2, separated by a distance a, orbiting each other. This could fx. be either a helium star or a sub-giant star orbiting a neutron star. We denote the two stars as the *donor* (M_2) and the *accretor* (M_1). The total mass, $M = M_1 + M_2$
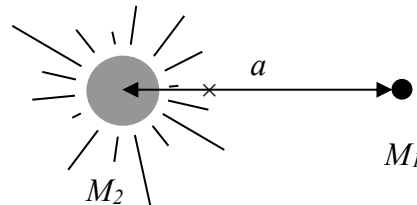
The orbital angular momentum and Kepler´s 3. law are given by:

$$J = \mu a^2 \Omega \qquad \wedge \qquad \Omega^2 = \frac{GM}{a^3} \qquad \text{where the reduced mass, } \mu = \frac{M_1 M_2}{M}$$

A simple combination of the three equations above yields: $\quad J^2 = G\dfrac{M_1^2 M_2^2}{M} a \qquad$ [1]

a) Use a logarithmic differentiation on the equation above and show:

$$\frac{\dot{a}}{a} = 2\frac{\dot{J}}{J} - 2\frac{\dot{M}_1}{M_1} - 2\frac{\dot{M}_2}{M_2} + \frac{\dot{M}_1 + \dot{M}_2}{M} \qquad [2]$$

In the following we neglect binary interaction effects such as magnetic braking, gravitational waves and tidal spin-orbit couplings and assume the only contribution to the loss of orbital angular momentum stems from wind mass loss. Furthermore, $\dot{M}_1 = 0$.
Assume a direct stellar wind carries off the specific orbital angular momentum of the donor. Thus the loss of orbital angular momentum from the system is given by:

$$dJ = \frac{J_2}{M_2} dM_2 \qquad\qquad [3]$$

where the specific orb.ang.mom. of the donor is: $\quad J_2 = \dfrac{M_1}{M} J \quad$ (can you derive this?)

Combining the two above expressions yields: $\quad \dfrac{\dot{J}}{J} = \dfrac{M_1}{M M_2} \dot{M}_2 \qquad\qquad$ [4]

b) Insert equation [4] into equation [2] and show $\dfrac{\dot{a}}{a} = -\dfrac{\dot{M}}{M}$ [5]

(remember $\dot{M}_1 = 0$)

c) Integrate the equation above and show: $\dfrac{M_i}{M_f} = \dfrac{a_f}{a_i}$ [6]

   where the indices "*i*" and "*f*" refer to the *initial* and *final* values, i.e. before and after the mass loss, respectively.

d) Show that the derived equation [6] can be expressed as $\dfrac{M_i}{M_f} = \sqrt{\dfrac{P_f}{P_i}}$ [7]

where P is the orbital period of the binary.

e) Consider a binary system consisting of a helium star of mass $3.3\,M_\odot$ and a neutron star of mass $1.4\,M_\odot$ orbiting each other with a period of 36.2 *days*. Assume the helium star loses mass in the form of a direct fast wind and after about 1 *Myr* has a mass of $2.9\,M_\odot$ before exploding in a supernova [Here we ignore the possibility of Case BB RLO]. Assume the neutron star does not accrete any of the material ejected from the helium star. Calculate the pre-SN orbital period.

f) Why does the orbit widen when orbital angular momentum is lost ?

g) Verify the change in orbital period in Fig.14.35 (Tauris & van den Heuvel, 2023) when the donor star emits a direct wind while evolving from the ZAMS until it begins RLO.

10. <u>Conservative mass-transfer</u>

In Roche-lobe overflow (RLO) the majority, sometimes all, of the mass lost by the donor star is transferred to the companion star. Assume *all* the mass lost by the donor star (M_2) is transferred and accreted onto the companion star (M_1), i.e. $\dot{M}_1 = -\dot{M}_2$

a) Consider eqn.[2] derived in question 9 and show: $\quad \dfrac{\dot{a}}{a} = 2\dfrac{\dot{J}}{J} - 2\left(1 - \dfrac{M_2}{M_1}\right)\dfrac{\dot{M}_2}{M_2}$

b) If orbital angular momentum is conserved then: $\quad \dot{J} = 0$ and $J_i = J_f$

Show: $\quad \dfrac{a_f}{a_i} = \left(\dfrac{M_{1i}\, M_{2i}}{M_{1f}\, M_{2f}}\right)^2 \qquad$ [8] $\qquad$ (hint: you can also use eqn.[1] in q.9)

c) Given that the total mass is also conserved during conservative mass-transfer combine the above equation with Kepler´s 3.law and derive:

$$\dfrac{P_f}{P_i} = \left(\dfrac{M_{1i}\, M_{2i}}{M_{1f}\, M_{2f}}\right)^3 \qquad \text{[9]}$$

An expression for the change in orbital separation in the general case where there are both direct wind mass loss and RLO, including mass loss from the accretor as well, is given in eqn.(4.58) in Tauris & van den Heuvel (2023).
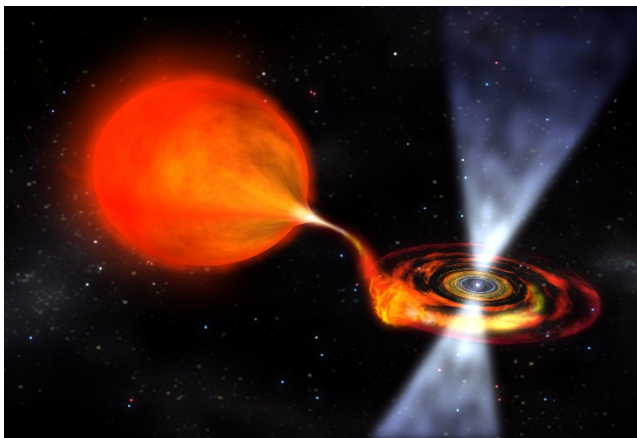
11. **<u>Common envelope evolution (CE)</u>**

Close binaries where the donor is much heavier than the accretor, or tight systems where the donor has a deep convective envelope, often end up in a common envelope after mass transfer has initiated. Due to viscous drag forces the captured accretor spirals in to the centre of the donor star, and as a result of the liberated orbital energy the donor star envelope is ejected. Let $0 < \alpha < 1$ (sometimes denoted by η) describe the efficiency of ejecting the envelope, i.e. of converting released orbital energy into kinetic energy providing the outward motion of the envelope: $E_{env} \equiv \alpha \, \Delta E_{orb}$ or

$$-\frac{G M_2 M_{2,env}}{\lambda \, a_i \, r_L} \equiv \alpha \left(-\frac{G M_{2,core} M_1}{2 \, a_f} + \frac{G M_2 M_1}{2 \, a_i} \right)$$

where $M_{2,core} = M_2 - M_{2,env}$; $r_L = R_L / a_i$ is the dimensionless Roche-lobe radius of the donor star so that $a_i \, r_L = R_L \approx R_2$, and λ is a parameter that depends on the mass-density distribution, and consequently also on the evolutionary stage of the donor (Dewi & Tauris 2000).

a) Use the equation above and derive: $\dfrac{a_f}{a_i} = \dfrac{M_{2,core} \, M_1}{M_2} \dfrac{1}{M_1 + 2 M_{2,env} / \alpha \, \lambda \, r_L}$

b) Verify the change in orbital separation (period) during the CE and spiral-in phase following the HMXB epoch in Fig.14.35 (Tauris & van den Heuvel, 2023) given that $r_L = R_L / a_i$ is found from their eqn.(4.6) and a modelling of the structure of the donor star at this epoch yields: $\alpha \cdot \lambda \approx 0.080$.

12. Fraction of all isolated stars ending as white dwarfs, neutron stars or black holes

The initial mass function (IMF) of stars in our Galaxy, $\Phi(M)\,dM$ is the number of stars formed per year per cubic parsec ($1\ pc = 3.08 \times 10^{18}\ cm$) within an interval of masses between M and $M+dM$. Salpeter (1955) found that:

$$\Phi(M)\,dM = 2 \times 10^{-12} \cdot M^{-2.35}\,dM \qquad stars\ yr^{-1}\ pc^{-3}$$

a) Estimate the fraction of all newborn stars which will end their lives as a WD, NS or BH. Assume progenitor ZAMS mass ranges of:
 $0.8 < M/M_\odot < 8$, $8 < M/M_\odot < 25$ and $25 < M/M_\odot < 150$, respectively.

b) i) Estimate the typical fraction of stellar material returned to the interstellar medium, ISM by progenitor stars of WDs, NSs and BHs, respectively.
   (Assume compact stellar masses of e.g.: $M_{WD} = 0.7 M_\odot$ $M_{NS} = 1.4 M_\odot$ $M_{BH} = 6.0 M_\odot$ and choose reasonable ZAMS masses of their progenitor stars within above intervals).

   ii) What would be the answer if the progenitor stars were located in close binaries?

The timescale needed for a star to exhaust its nuclear fuel reserve is roughly given by:

$$\tau_{nuclear} = 10 \cdot \left(M_{ZAMS}/M_\odot\right)^{-2.5}\ Gyr$$

c) What are typical timescales for the lifetime of the progenitor stars of a WD, NS or BH?
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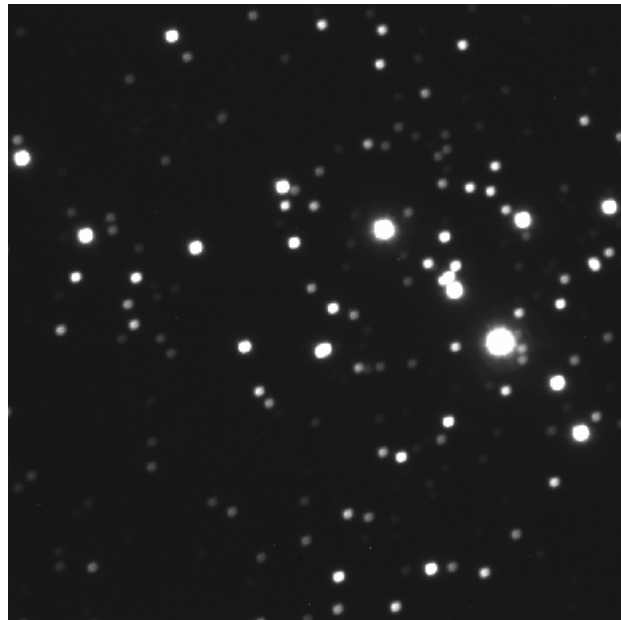
13. <u>Energetics of accreting objects</u>

In our Galaxy there are about 100 bright X-ray sources with fluxes well above 10^{-10} erg cm^{-2} s^{-1} in the energy range 1-10 keV. These fluxes correspond to source luminosities of $L_X \approx 10^{34} - 10^{38}$ $erg\ s^{-1}$. These luminosities are roughly equivalent to the rate at which gravitational potential energy is released from matter falling down the deep potential well of a compact object (see also Table 6.1 in Tauris & van den Heuvel, 2023).

a) Calculate the accretion rate ($M_\odot\ yr^{-1}$) needed to feed a $1.4\,M_\odot$ neutron star with a detected X-ray flux, $F_X = 2.8 \times 10^{-9}\ erg\ cm^{-2}\ s^{-1}$ located at a distance of 7.0 kpc.

b) Does this accretion rate exceed the Eddington limit for a neutron star ?
(see e.g. Section 4.9 in Tauris & van den Heuvel, 2023).
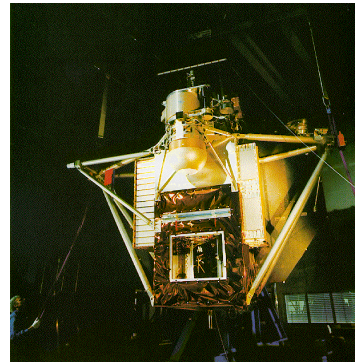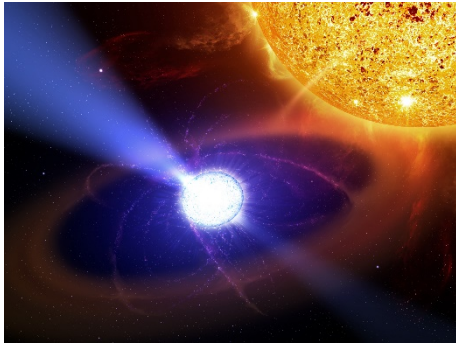
**14. <u>Thermal emission from a nearby neutron star</u>**

For many years, the closest known neutron star was considered to be PSR J0108-1431 (Tauris et al. 1994) which is estimated to be only 85 pc from the Solar system. It has a characteristic age of 160 Myr. However, if the surface magnetic field decays on a short time-scale, the true age of the pulsar is much younger. Fx. if τ_D = 5 *Myr* then t ≈ 10 *Myr* (see q.4). Since neutron stars cool down with age the temperature, and thus the wavelength of thermal (black body) emission from its surface, could help to set constraints on the decay time-scale of the B-field. Some cooling models predict a temperature of 125 000 *K* for a neutron star age of 10 *My*r.

The ROSAT all-sky survey operated within the soft X-ray band (0.1-2.4 *keV*) and detected sources down to a limiting flux of 5×10^{-13} *erg cm*$^{-2}$ *s*$^{-1}$.

a) Find the wavelength (and the photon energy) of maximum brightness (intensity) if indeed the pulsar is 10 *Myr* old. Is it within the energy band of ROSAT ?

b) Calculate the expected flux detected from the pulsar and once again compare with the ROSAT survey.

c) What kind of observation would you propose for PSR J0108-1431 in order to help determining its true age (via cooling models) and thus setting a constraint on the time-scale for B-field decay ?

15. <u>Hydrogen fusion versus accretion onto a neutron star</u>

a) Fusion of hydrogen ($H \rightarrow He$)

    (i) Calculate the amount of energy liberated in the fusion of 1 kg of hydrogen

    (ii) Calculate the percentage of energy liberated during hydrogen fusion in units of the rest mass energy of the hydrogen used

b) Accretion onto a neutron star (assume: $R_{NS} = 10$ km., $M_{NS} = 1.4 M_\odot$)

    (i) Estimate the amount of (gravitational potential) energy liberated when 1 litre of water is poured onto a neutron star from far away

    (ii) Calculate the percentage of energy liberated during the accretion process in units of the rest mass energy of the infalling matter
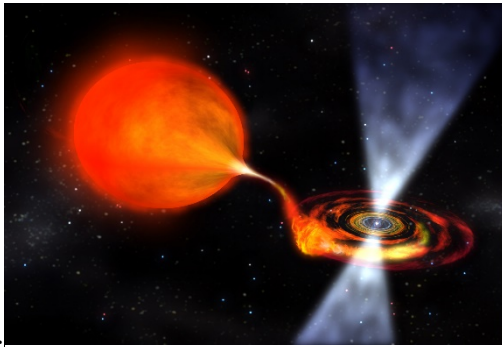
c) Answer question b) above for the Sun, a WD and a BH.

16. X-ray emission from an accreting black hole

a) Show that the accretion luminosity of an object accreting matter with a rate of

$$\dot{M} = \frac{dM}{dt} \text{ is given by: } L_{acc} = \frac{G M \dot{M}}{R}$$

b) A former X-ray satellite called ROSAT had a limiting detection threshold flux of $F_X \approx 5 \times 10^{-13} \quad erg \; cm^{-2} \; s^{-1}$. Consider an accreting black hole at a distance of 1 kpc. Calculate the accretion rate needed in order for ROSAT to detect this X-ray source assuming $M_{BH} = 4 M_{\odot}$ (note: the Schwarzchild radius of a black hole is $R_{Sch} = \frac{2 G M}{c^2}$)

   i) units of grams per second
   ii) in units of $M_{\odot} \; yr^{-1}$

c) Discuss the assumption of using: $F_x = \frac{L_{acc}}{4 \pi d^2}$ (hint: SED)

and whether or not $R = R_{Sch}$ is a good approximation when estimating the released gravitational binding energy of the accreted matter.
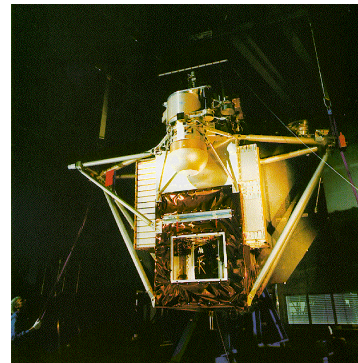
17. <u>Supernova explosion energetics</u>

a) Estimate the energy release in the gravitational collapse of an iron core of mass $1.4\,M_\odot$ from an initial density of $10^9\ g\,cm^{-3}$ (typical density in the silicon burning central region of a massive star) to a final density of nuclear matter: $2.8 \times 10^{14}\ g\,cm^{-3}$.
Assume the stellar core is a uniform, homogeneous sphere, *i.e.* $\rho(r)$ is constant.

b) Compare the result to the rest-mass energy of the core.

The gravitational mass of a NS (as measured by a distant observer by its gravitational effects) contains not only the rest-mass of the baryons, but also the (negative) mass-equivalent of the binding energy, $\Delta M_{def} = E_{bind}/c^2 < 0$. This is also the max efficiency of radiative emission in units of the available rest-mass energy incident on an accreting NS according to the virial theorem.

The maximum luminosity following the energy release from a supernova explosion is comparable to the luminosity of an entire galaxy ($10^{11}\ L_\odot$) for several days.
Even so, only a tiny fraction of the total energy release from the gravitational collapse is emitted in the form of photons. Neutrinos escape with 99% of the energy output.

c) Estimate the number of days a SN is seen to outshine a galaxy
Assume 1% of the energy released in the SN goes into photons. $L_\odot = 3.846 \times 10^{33}\ erg\ s^{-1}$.

The cross section for interaction between a neutrino and a neutron is assumed to be:
$$\sigma_{vn} = 10^{-44}\ cm^2.$$

d) What is the probability that a neutrino generated at the centre of a neutron star will escape unimpeded? (Assume here R_{NS} = 10 km)
Hint: The probability that the neutrino travels a distance r without interacting with a neutron is given by: $P(r) \approx e^{-r/\lambda}$ where λ is the mean free path of the neutrino.
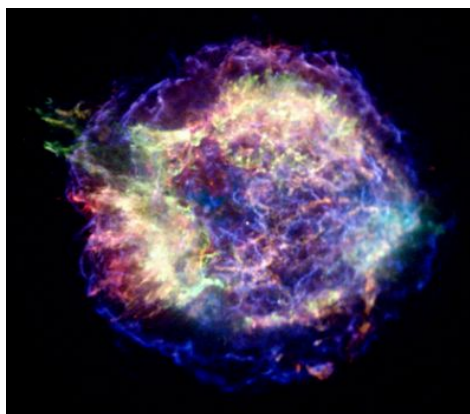
18. Effects of supernova explosions in binaries

Consider a supernova explosion of a helium star of mass $3.9\,M_\odot$ in a binary system with a $1.6\,M_\odot$ non-deg. companion star. The explosion results in a $1.3\,M_\odot$ neutron star. Assume the pre-SN orbital period is 3.5 days.

a) Show that Keplers 3.law: $\left(\dfrac{2\pi}{P}\right)^2 = G\dfrac{M}{a^3}$, where $M = M_1 + M_2$, can be rewritten as:

$$P = \frac{1}{8.626}\cdot\sqrt{\frac{(a/R_\odot)^3}{M/M_\odot}} \qquad \text{days} \qquad\qquad [1]$$

and calculate the pre-SN orbital separation in units of $R_\odot$.

b) Assume the supernova explosion is *symmetric* ($w = 0$) and use eqn. (13.3) in Tauris & van den Heuvel (2023) to calculate the post-SN orbital separation and also the post-SN orbital period.

c) Combine eqns.(3.4), (13.5) and (13.6) to show: $e = 1 - \dfrac{a_0}{a}$ \qquad [2]

for a symmetric SN (i.e. no kick, $w=0$) and calculate the post-SN eccentricity.

(Note, $\mu = (M_1\cdot M_2 / (M_1 + M_2))$ and v_{rel} after the SN is equal to the pre-SN v_{rel}). (See also Exercise 4.10 in Tauris & van den Heuvel, 2023).
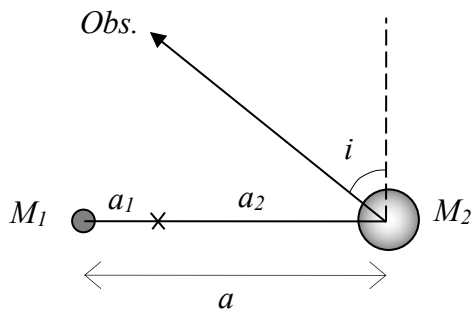
d) Calculate the post-SN periastron distance between the neutron star and its companion.

e) Use eqns.(13.12)–(13.13) to show that the binary system receives a recoil speed: $v_{sys} = 65\,km\,s^{-1}$.

(Note, to find the speed of the exploding He-core use: $M_1\cdot v_1 = M_2\cdot v_2 = \mu\cdot v_{rel}$)

19. Constraining masses of compact stars in binaries

a) Consider the circular binary (illustrated above) with a compact object (M_1) and a companion star (M_2). Use conservation of momentum to show that: $\mu a = M_1 a_1 = M_2 a_2$

where $\mu = \dfrac{M_1 \cdot M_2}{M_1 + M_2}$ is the reduced mass.

b) The mass function of a binary system is defined by: $f(M_1, M_2, i) \equiv \dfrac{(M_1 \cdot \sin i)^3}{(M_1 + M_2)^2}$.

Show that $f(M_1, M_2, i) \equiv \dfrac{(M_1 \cdot \sin i)^3}{(M_1 + M_2)^2} = \dfrac{P \cdot v_2^3}{2\pi G}$ where $v_2 = \dfrac{2\pi a_2}{P} \cdot \sin i$ is the

projected orbital velocity of the companion star (parallel to the line of sight between the observer and the binary) and P is the orbital period of the binary given by Kepler´s 3. law.

c) The Galactic black hole Cyg X-1 has an orbital period of 5.6 days and a measured mass function, $f = 0.25\, M_\odot$. Spectral analysis of the companion by Miller-Jones et al. (2021) yields a companion star mass, $M_2 = 41\, M_\odot$. Modern EOS suggest a maximum NS mass of $M_{NS}^{max} = 2.5\, M_\odot$. Verify that Cyg X-1 is indeed a black hole binary system.

d) Assuming the inclination angle, i of the binary is:

 (i) 60° calculate the black hole mass.
 (ii) 20° calculate the black hole mass.
 (iii) 27.5° as measured by Miller-Jones et al. (2021), show that $M_{BH} = 21\, M_\odot$

e) Show that the median inclination angle is 60°.

20. **Degenerate Fermi gas**
   a) Show that the number density of a degenerate Fermi gas is given by:
      $$n = p_F^3 / (3\pi^2 \hbar^3)$$
   b) Rewrite the expression by introducing the Compton wavelength: $\lambda \equiv \hbar / mc$ and the relativity parameter: $x \equiv p_F / mc$
   c) Express ρ (in g/cm³) as a function of x in the case of:
      I. an electron gas $(\rho = n_e \cdot \mu_e \cdot m_u)$ - use $\mu_e \simeq 2$ which is a typical value for a CO WD
      II. a pure neutron gas $(\rho = n_n \cdot m_n)$

We notice that in the core of a WD ($\rho \sim 10^9 \ g \, cm^{-3}$) we have an ext. rel. deg. electron gas and in the outer parts of the WD ($\rho \sim 10^5 \ g \, cm^{-3}$) we have a non-rel. deg. electron gas. For a NS ($\rho \sim 10^{15} \ g \, cm^{-3}$) we only need to consider a non-rel. deg. neutron gas.

Checking the validity of degeneracy.
In order to lift degeneracy (and obtaining a classical Maxwell-Boltzmann gas), the temperature must be above the Fermi temperature, i.e.: $T > T_F$ where $k_B T_F = E_F$.
   d) Calculate this threshold temperature, T_{deg} for a WD (non-rel. and ext. rel. electron gas) as well as for a NS (non.-rel. neutron gas). Use the mass densities stated above.
   e) Compare with observations.
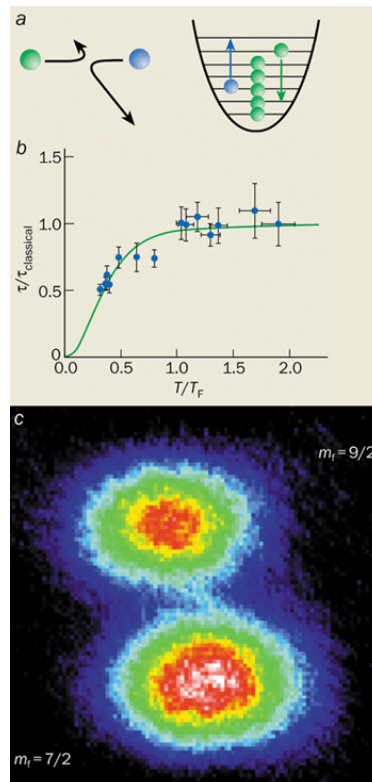
21. **Polytropic degenerate gas**

a) Consider a non-rel. deg. gas ($E_F = p_F^2 / 2m$)

   (i) Show that the (internal) energy density is given by: $u = \dfrac{3}{5} n E_F$

   (ii) Given that the pressure: $P = \dfrac{2}{3} u$, show: $P = \dfrac{(3\pi^2 \hbar^3)^{2/3}}{5m} n^{5/3}$

b) Consider an ext. rel. deg. gas ($E_F = p_F c$)

   (i) Show that the (internal) energy density is given by: $u = \dfrac{3}{4} n E_F$

   (ii) Given that the pressure: $P = \dfrac{1}{3} u$, show: $P = \dfrac{c(3\pi^2 \hbar^3)^{1/3}}{4} n^{4/3}$

A simple equation of state (EOS) for compact objects is based on a polytropic model of a degenerate gas: $P = K \rho^{\Gamma}$, where $\Gamma \equiv \left. \dfrac{\partial \ln P}{\partial \ln \rho} \right|_S$ is the adiabatic exponent.

c) Derive the value(s) of K (in cgs-units) for a WD and a NS.

In stellar structure theory one uses the Lane-Emden equation to find the structure of a polytropic star. One can show that: $K = N_n G M^{(n-1)/n} R^{(3-n)/n}$ where

$n \equiv \dfrac{1}{\Gamma - 1}$ is the so-called polytropic index.

One can also show:

$$N_n = \begin{cases} 0.42422... & \Gamma = 5/3, \quad n = 3/2 \quad \text{non-rel. gas} \\ 0.36394... & \Gamma = 4/3, \quad n = 3 \quad \text{ext. rel. gas} \end{cases}$$

d) Derive an EOS for a WD and a NS, e.g.
   - state R (in units of $R_\odot$) as a function of ($M / M_\odot$) for a WD with $\Gamma = 5/3$
   - state R (in units of km) as a function of ($M / M_\odot$) for a NS with $\Gamma = 5/3$
   - state the critically unstable ($E \simeq 0$) Chandrasekhar mass limit of a WD with $\Gamma = 4/3$ (note: this mass limit is independent of radius).

e) Find the radius of a $0.7\, M_\odot$ WD and a $1.4\, M_\odot$ NS
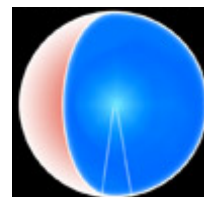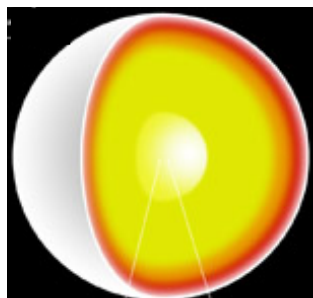
22. Inverse β-decay (n,p,e)-gas

    a)   Go through eqs.(2.5.1) to (2.5.13) in Shapiro & Teukolsky

    b)   Consider a WD with positive ions and an e-gas (mean molecular weight $\mu_e \simeq 2$).
        Estimate the threshold mass density for the onset of Pauli blocking, in the case of:

          (i)   A non-rel. deg. electron gas
         (ii) An ext. rel. deg. electron gas

    c)   Derive eqs.(2.5.16) and (2.5.17) from eqn.(2.5.15) and show

$$\left(\frac{n_p}{n_n} \right) \rightarrow \frac{1}{8} \quad for \quad x_n \rightarrow \infty$$

        Note, in the book: $Q = m_n - m_p$ and not $(m_n - m_p) \cdot c^2$.
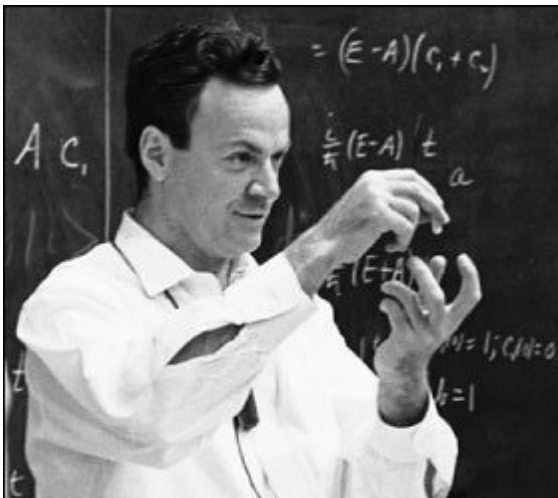
23. Cooling of white dwarfs

a) Starting with the photon diffusion eqn.(4.1.1) in Shapiro & Teukolsky derive eqs.(4.1.2), (4.1.5) and (4.1.7). Before deriving eqn.(4.1.7) find $P(T)$.

b) At the core boundary, assume $P_{gas} = P_{deg}$ (use a non-rel. deg. electron gas polytrope) and derive an expression for ρ at the core boundary (ρ_*) which can then be combined with eqn.(4.1.7) to yield: $L = C \cdot M \cdot T_{core}^{7/2}$. Show this derivation.

c) Go through eqs.(4.2.1) to (4.2.8) in chapter 4.2 and show that the cooling age (the time since the formation of the white dwarf) is given by: $\tau \propto \left(\dfrac{M}{L} \right)^{5/7}$

d) Calculate the cooling time of a WD with $M = 0.7 \, M_\odot$ and $L = 10^{-3} \, L_\odot$
(Hint: you´ll need eqns.(4.1.11) and (4.2.7))
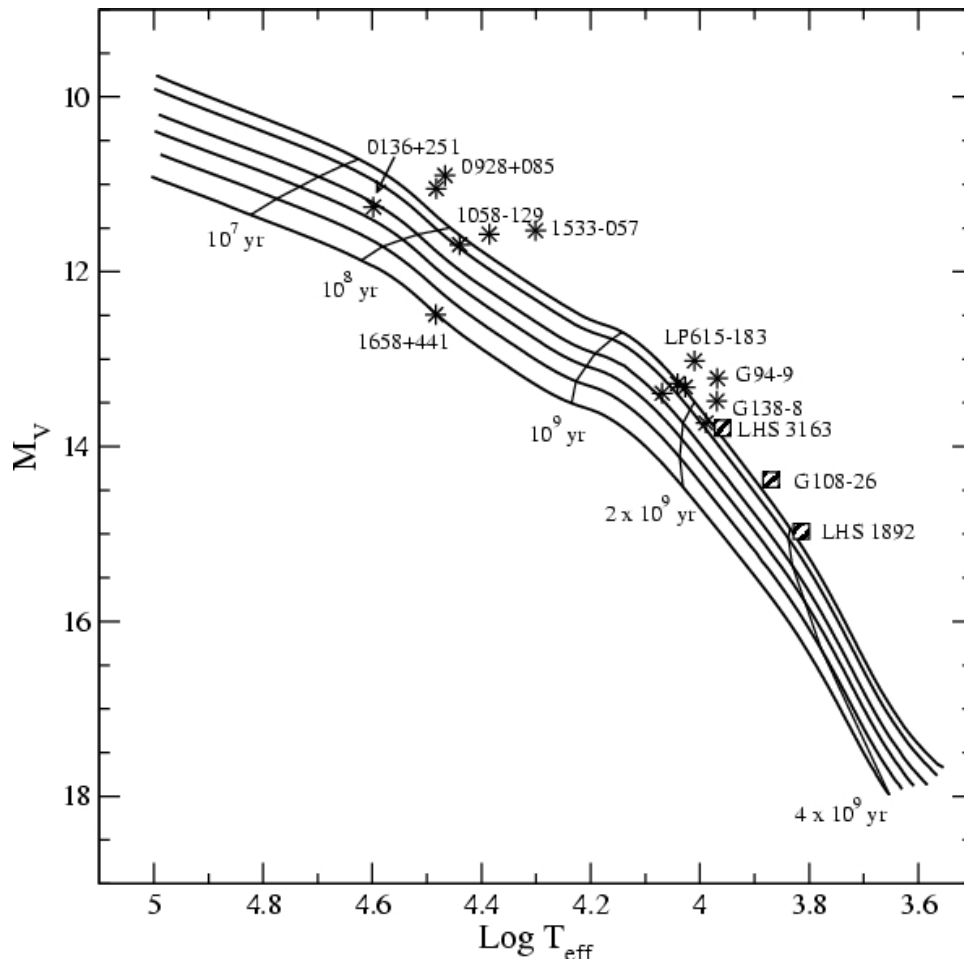
24. Cooling ages of white dwarfs

a) Consider a pure carbon WD with a mass of $0.7\,M_\odot$ and an interior temperature of
   $T = 10^7\,K$.

 (i) Find its total thermal energy and its luminosity.
     (You may as a first try ignore the dependence on chemical composition (X, Y, Z))
 (ii) Assume (not correct!) that its luminosity stays constant while it cools off.
     For how long time will the WD shine?

b) Consider another (very young) WD assumed to be composed of ionized ^{12}C.
   (Assume that its surface composition is (X, Y, Z) = (0.0, 0.90, 0.10)).
   Observations of spectral lines yield a surface temperature, $T_{eff} = 90\,000\,K$. The mass of the
   WD is estimated to be $0.8\,M_\odot$.

 (i) What is the (surface) luminosity of the WD? (<u>Hint</u>: use the $R(M)$-relation from q.21d)
 (ii) Calculate the interior temperature of the WD.
 (iii) Is neutrino emission important?

 (iv) Is the approximation of an interior degenerate electron gas valid? (cf. q.20d and use $\Gamma = \dfrac{4}{3}$)

 (v) What is the cooling age of this WD? (assuming cooling only by emission of photons)

c) Exercise 4.1 in Shapiro & Teukolsky

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