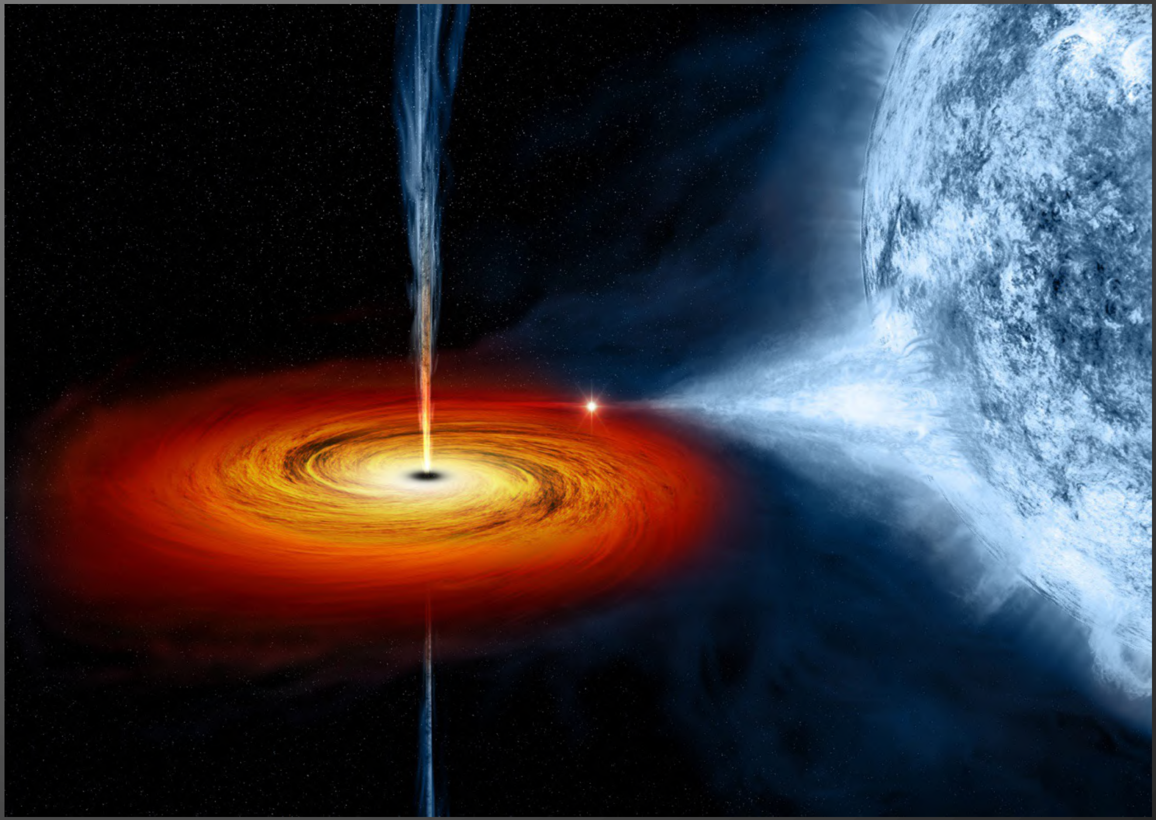
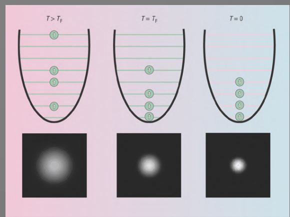
PHYSICS OF COMPACT OBJECTS
AND THEIR BINARY INTERACTIONS

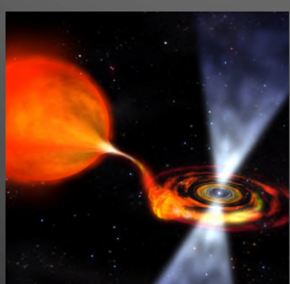


AALBORG UNIVERSITY

Thomas Tauris – Physics, Aalborg University

Programme





- ∗ **Introduction**
- ∗ **Degenerate Fermi Gases**
  Non-relativistic and extreme relativistic electron / (n,p,e^-) gases
- ∗ **White Dwarfs**
  Structure, cooling models, observations
- ∗ **Neutron Stars**
  Structure and equation-of-state
- ∗ **Radio Pulsars**
  Characteristics, spin evolution, magnetars, observations, timing
- ∗ **Binary Evolution and Interactions**
  X-ray binaries, accretion, formation of millisecond pulsars, recycling
- ∗ **Black Holes**
  Observations, characteristics and spins
- ∗ **Gravitational Waves**
  Sources and detection, kilonovae
- ∗ **Exam**

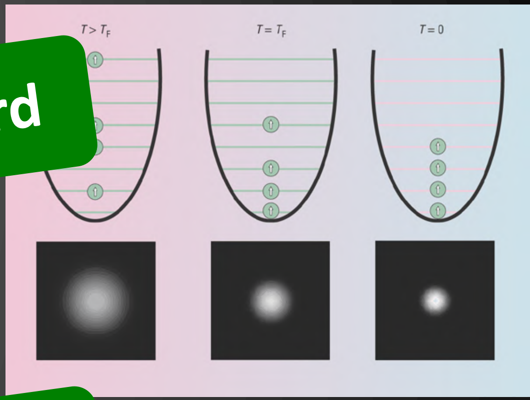
Degenerate Fermi Gases

and applications to simple EoS

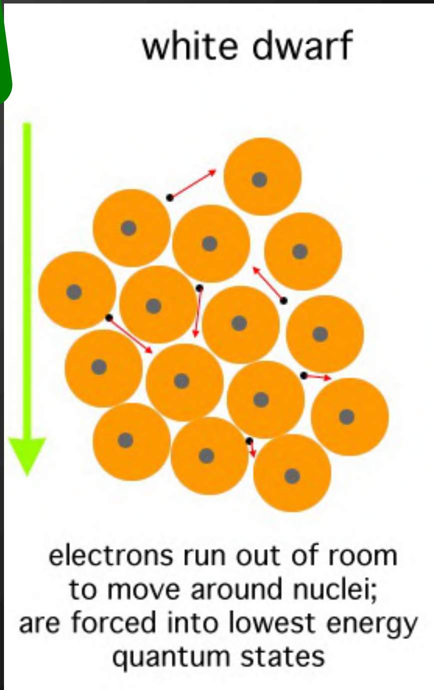
- Degenerate matter
- Kinetic theory
  - Distribution function in 6-dim phase-space
  - Fermi-Dirac statistics
  - Number density (n), energy density (u), pressure (P)
- Complete degenerate ideal Fermi gas ($T \rightarrow 0$)
  - Non-relativistic gas
  - Relativistic gas
  - The question of relativity and degeneracy
- Polytropic EoS
  - R(M) relations for a WD (electron gas) and a NS (neutron gas)
- Three important corrections:
  - Electrostatic corrections
  - Inverse beta-decay
  - General relativity (for NSs)

Blackboard

Blackboard



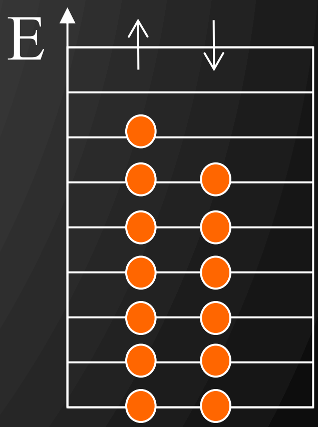
(Chandrasekhar 1931)



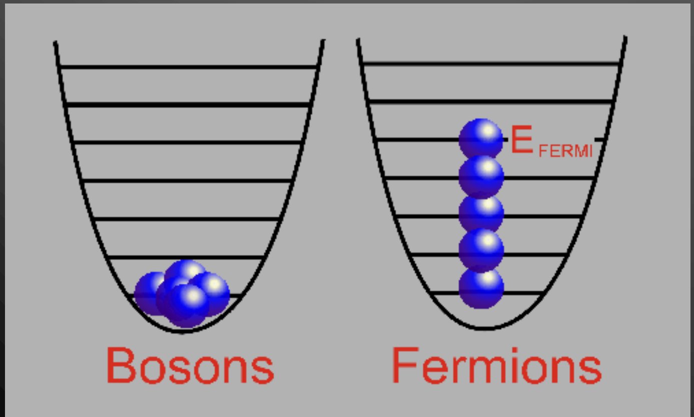
Degenerate gas

Quantum mechanics is important for the description of a gas at low temperature and high density.

Degenerate matter is matter which has sufficiently high density that the dominant contribution to its pressure (called *degeneracy pressure*) rises from the Pauli exclusion principle which forbids the constituent particles to occupy identical quantum states (i.e. two particles cannot be in the exact same place at the exact same time).



This forces particles into states with higher energies which increases the internal energy and pressure to values beyond the normal values for an ideal gas.

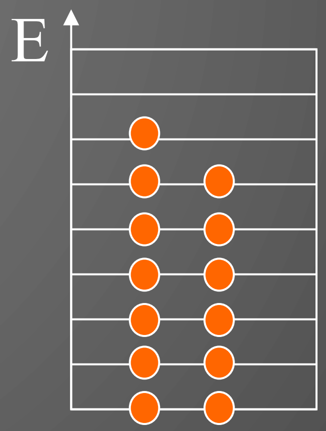


Only femions (odd half-integer spin particles) obey the Pauli exclusion principle. The reason for this is not well understood….
Bosons (integer spin particles) form Bose-Einstein condensates instead.

Degenerate gas

Pauli exclusion principle analogy

Degenerate gas:
low temperature and high density
→ high energy states occupied



Manhattan:
small area and high demand
→ skyscrapers

Degenerate gas

In particular $\lim_{T \to 0} U \neq 0 \qquad \lim_{T \to 0} P \neq 0$

(i.e. finite internal energy and finite pressure for T→ 0).
Degeneracy pressure does not depend on temperature!

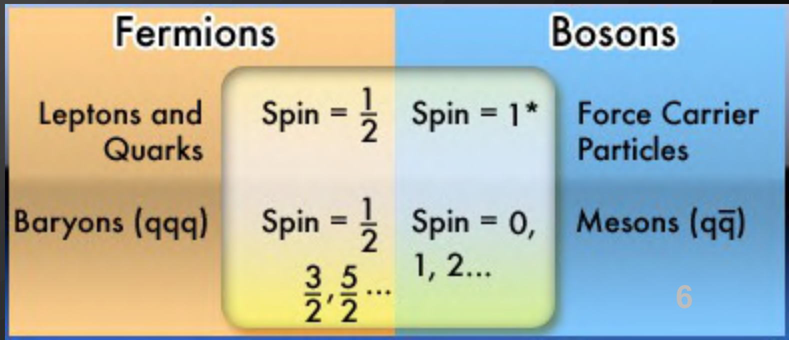
$$\cancel{P_{\rm deg}(T)} \qquad \wedge \qquad P_{\rm ideal\ gas} \propto T$$

Below we shall study the effect of a degenerate Fermi gas
(which is a gas made up of *fermions*)

Fermions:
24 elementary fermions with half-integer spin
+ composite fermions (odd half-integer spin)
→ e.g. protons, neutrons

baryons

6 quarks + 6 anti-quarks (u, d, s, c, t, b)
6 leptons + 6 anti-leptons (e^-, μ^-, τ^-, ν_e, ν_μ, ν_τ)



Fermions		Bosons	
Leptons and Quarks	Spin = $\frac{1}{2}$	Spin = 1*	Force Carrier Particles
Baryons (qqq)	Spin = $\frac{1}{2}$, $\frac{3}{2}$, $\frac{5}{2}$...	Spin = 0, 1, 2...	Mesons ($q\bar{q}$)

Fermi-Dirac statistics

Blackboard

Kinetic theory distribution of fermions in phase space

Distribution function in 6-dim. phase space *f* (***x***,***p***,*t*):

$$\frac{dN}{d^3x\, d^3p} = \frac{g}{h^3} f$$

Provides a full description of the system

$$\frac{dN}{d^3x\, d^3p} = \frac{g}{h^3} f$$

$g = 2S + 1$ statistical weight

$S = ½$ (n, p, e$^-$) → $g = 2$ spin ↑↓

occupation number
(probability of finding a fermion in a given state)

average # particles in a Planck cell (h^3)

volume of a Planck cell in phase space

number density (in real space) of each species of particles is given by:

$$n = \int \frac{dN}{d^3x\, d^3p} d^3p = \frac{2}{h^3} \int f(p)\, d^3p$$

integrating over momentum space

$$u = \int E(p) \frac{dN}{d^3x\, d^3p} d^3p = \frac{2}{h^3} \int E(p) f(p)\, d^3p$$

energy density

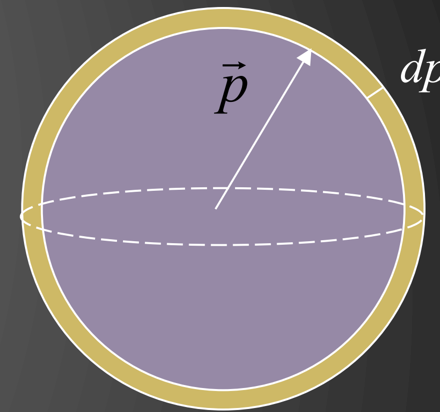
Fermi-Dirac statistics

$$n = \frac{2}{h^3} \int f(p)\, d^3 p$$

$$\Downarrow$$

$$n = \frac{2}{h^3} \int f(p)\, 4\pi p^2\, dp$$

$$\wedge \quad f(p) = \frac{1}{\exp((E-\mu)/k_B T) + 1}$$



momentum space

no analytical solution

energy of particle with momentum p

$$E = \sqrt{(m_0 c^2)^2 + (pc)^2}$$

chemical potential

$$\mu \equiv \frac{\partial u}{\partial n}$$

Complete degeneracy

limit of $T \rightarrow 0$:

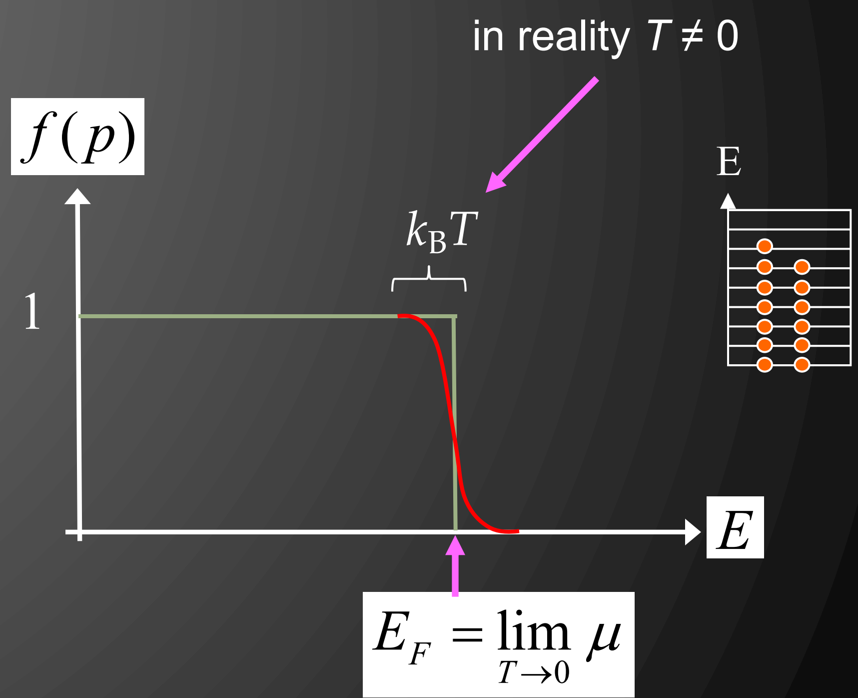
$$f(p) = \frac{1}{\exp((E-\mu)/k_B T)+1}$$

$$f(p) \rightarrow \begin{cases} 1 & E < E_F \\ 0 & E > E_F \end{cases}$$

$$\Downarrow$$

$$n = \frac{2}{h^3}\int_0^{p_F} 4\pi p^2 dp = \frac{p_F^3}{3\pi^2\hbar^3}$$

Fermi momentum



Fermi energy

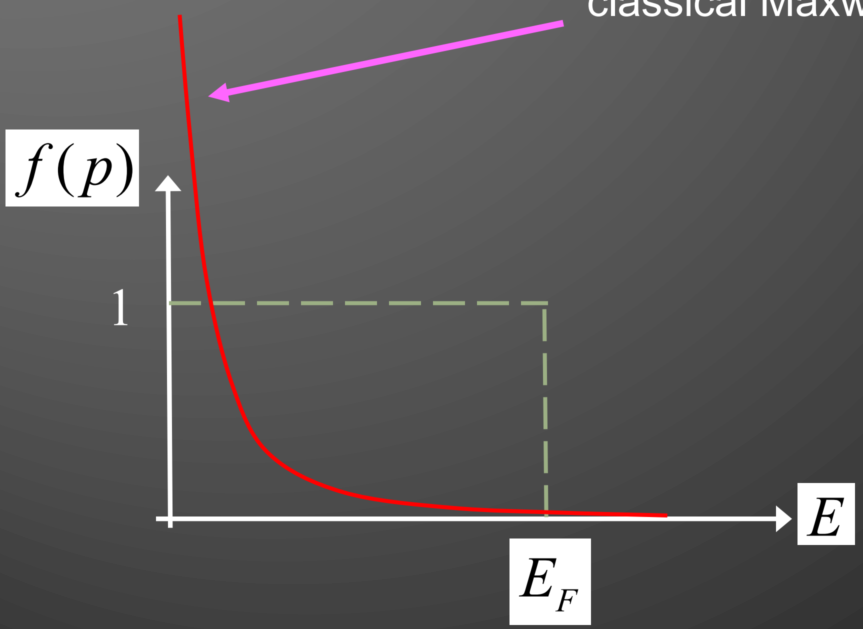
Non-degeneracy classical limit

for low densities and high temperatures:

$$f(p) = \frac{1}{\exp((E-\mu)/k_B T)+1} \quad \rightarrow \quad \exp((\mu - E)/k_B T)$$

non-trivial b/c $\mu(T)$

classical Maxwell-Boltzmann distribution!



Complete degeneracy

non-relativistic particles:

$$E = p^2 / 2m$$

$$u = \frac{3}{5} n E_F$$

$$P = \frac{2}{3} u \propto n \cdot n^{2/3} \propto \rho^{5/3}$$

extremely relativistic particles:

$$E = pc$$ energy of particle

$$u = \frac{3}{4} n E_F$$ energy density of gas

$$P = \frac{1}{3} u \propto n \cdot n^{1/3} \propto \rho^{4/3}$$ pressure

$$P = K \rho^{\Gamma}$$

polytropic EoS

relativistic gas in WD and NS?

Relativistic Fermi gas?

Ex.20

$$\lambda_i \equiv \frac{\hbar}{mc}$$ Compton wavelength

$$\chi_i \equiv \frac{p_F}{mc}$$ relativity parameter

$$\chi_i = \begin{cases} \ll 1 & \text{for non-rel. particles} \\ \gg 1 & \text{extremely rel. particles} \end{cases}$$

$$n = \frac{1}{3\pi^2 \lambda_i^3} \chi_i^3$$

electron gas:
(white dwarf)

mass density

$$\rho = n_e \mu_e m_u = 2\times 10^6 \ \chi_i^3 \ \text{g cm}^{-3}$$

$$\mu_e = 2$$

$$\text{WD} \begin{cases} \rho = 10^5 \ \text{g cm}^{-3} & \text{envelope} & \text{non-rel.} \\ \rho = 10^9 \ \text{g cm}^{-3} & \text{core} & \text{ext. rel.} \end{cases}$$

WD: non-rel. *and* ext. rel. e-gas

neutron gas:
(neutron star)

$$\rho = n_n m_n = 6\times 10^{15} \ \chi_i^3 \ \text{g cm}^{-3}$$

$$NS \begin{cases} \rho_{nuc} = 2.8\times 10^{14} \ \text{g cm}^{-3} & \text{nuclear} & \text{non-rel.} \\ \rho = 10^{15} \ \text{g cm}^{-3} & \text{core} & \text{non-rel.} \end{cases}$$

NS: *only* non-rel. n-gas

Degeneracy in WDs and NSs?

Ex.20

Degeneracy is lifted in case: $k_B T_* \geq E_F$

$$E_F = \begin{cases} p_F^2 / 2m & \text{for non-rel. particles} \\ p_F c & \text{for extremely rel. particles} \end{cases}$$

$$n = \frac{p_F^3}{3\pi^2 \hbar^3}$$

electron gas:
(white dwarf)

$$\rho = n_e \mu_e m_u$$

$$\text{WD} \quad \begin{cases} \rho = 10^5 \text{ g cm}^{-3} & \text{envelope} & \text{non-rel.} & T_* \geq 10^8 \ K \\ \rho = 10^9 \text{ g cm}^{-3} & \text{core} & \text{ext. rel.} & T_* \geq 10^{10} \ K \end{cases}$$

neutron gas:
(neutron star)

$$\rho = n_n m_n$$

$$NS \quad \rho = 10^{15} \text{ g cm}^{-3} \quad \text{core} \quad \text{non-rel.} \quad T_* \geq 10^{12} \ K$$

These critical temperatures, T_* exceed the interior temperatures of WDs and NSs (except in very early phases), hence these objects are indeed made of degenerate matter

Polytropic equation-of-state

Ex.21

adiabatic index

$$P = K\rho^{\Gamma}$$ (1), need to calculate K and Γ

$$\Gamma = \frac{\partial \ln P}{\partial \ln \rho} \quad \wedge \quad n \equiv \frac{1}{\Gamma - 1}$$

polytropic index

$$\Gamma = \begin{cases} 5/3 \quad n=3/2 \quad \text{for non-rel. particles} \\ 4/3 \quad n=3 \quad \text{for extremely rel. particles} \end{cases}$$

Solve the Lane-Emden eqn:

for the structure of a polytrope of index *n*.

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) = -\theta^n, \quad \text{where} \quad \begin{cases} r = \alpha\,\xi \\ \rho = \rho_c \theta^n \end{cases}$$

measure of distance to centre

$$\theta = \theta(\xi)$$

(a dimensionless form of Poisson's eqn.
for the grav. pot. of a Newtonian, self-gravitating, spherically symmetric, polytropic fluid).

Integration yields:

$$K = N_n G M^{(n-1)/n} R^{(3-n)/n}, \quad \text{where} \quad N_n = \begin{cases} 0.42424\ldots \quad n = 3/2 \\ 0.36394\ldots \quad n = 3 \end{cases}$$ (2)

Combining (1)** and (2) yields:

non-rel.:

$$\text{WD}: \quad R = 0.013\, R_\odot\, (M/M_\odot)^{-1/3}$$
$$\text{NS}: \quad R = 15.4\ \text{km}\, (M/M_\odot)^{-1/3}$$

ext. rel.:

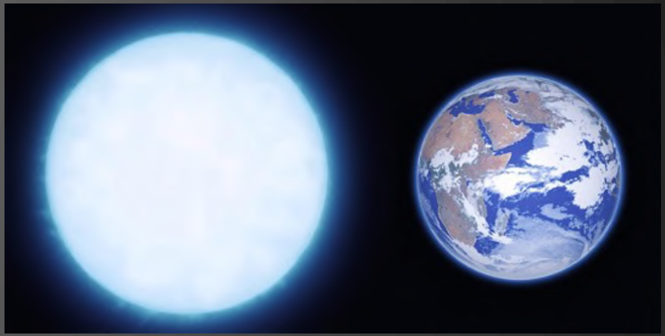
$$\text{WD}: \quad M = 1.457\, M_\odot$$

Chandrasekhar mass limit (1931)

** $P(u) \ \wedge \ u(n, E_{\mathrm{F}}) \rightarrow P(n) \ \wedge \ \rho = n\mu_e m_u \rightarrow P(\rho) \rightarrow K$

Polytropic equation-of-state

$$\mathrm{WD:}\quad R = 0.013\ R_\odot\ (M/M_\odot)^{-1/3}$$
$$\mathrm{NS:}\quad R = 15.4\ \mathrm{km}\ (M/M_\odot)^{-1/3}$$



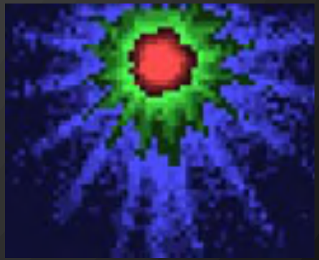
Example:

$$\mathrm{WD:}\quad M = 0.7\ M_\odot \Rightarrow \quad R = 10200\ \mathrm{km}$$
$$\mathrm{NS:}\quad M = 1.4\ M_\odot \Rightarrow \quad R = 13.8\ \mathrm{km}$$

Surprisingly good!!

Observational tests:

$$\frac{\Delta\lambda}{\lambda} = \frac{GM}{Rc^2}$$ Gravitational redshift of WDs

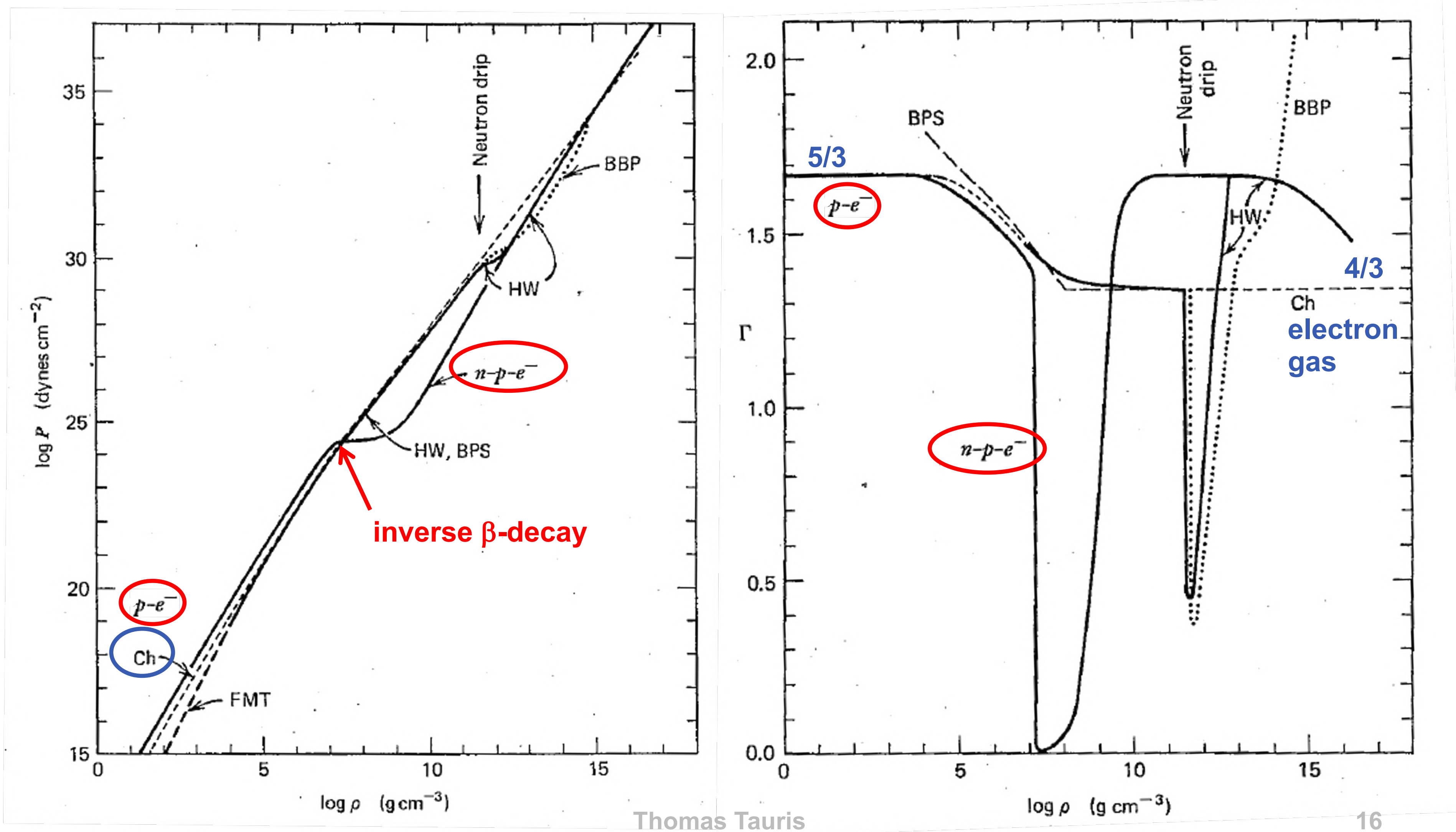


$$\log g = \frac{GM}{R^2}$$ Surface gravity of WDs

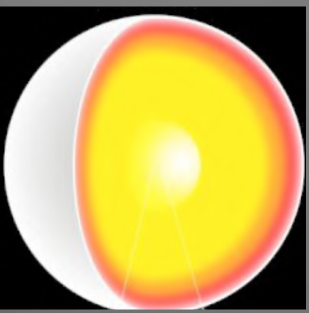
$$L = 4\pi R^2 \sigma T^4$$ Block body fitting to X-ray spectra of young, hot NSs

Degenerate Fermi Gases

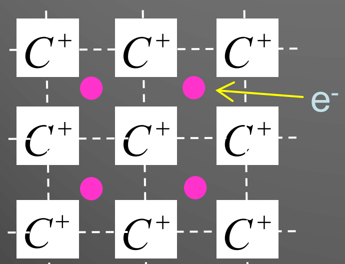
and applications to simple EoS



Three important corrections to the polytropic equation-of-state



WD ions in lattice with electron gas



Wigner-Seitz cell approximation

WDs are not made up of a pure electron gas (Chandrasekhar EoS)
NSs are not made up of a pure neutron gas

- Inverse β-decay (n, p, e$^-$) gas - see next slide
- Electrostatic corrections (electrons interact with ion nuclei)

$$E_{coulomb} < 0 \quad \Rightarrow \quad P\downarrow \quad \Rightarrow \quad \text{softer EoS} \quad \left(R\downarrow, \rho_c\uparrow\right)$$

$$\frac{E_{coulomb}}{E_{kin}} = \frac{Z\,e^2 / \langle r \rangle}{k_B T}$$

- GR effects: make NSs more unstable b/c strong self-gravity

$$\text{Stability criterion:} \quad \bar{\Gamma} > \frac{4}{3} + \kappa \frac{GM}{Rc^2}$$

Inverse β-decay and Pauli blocking

(p, e⁻) gas

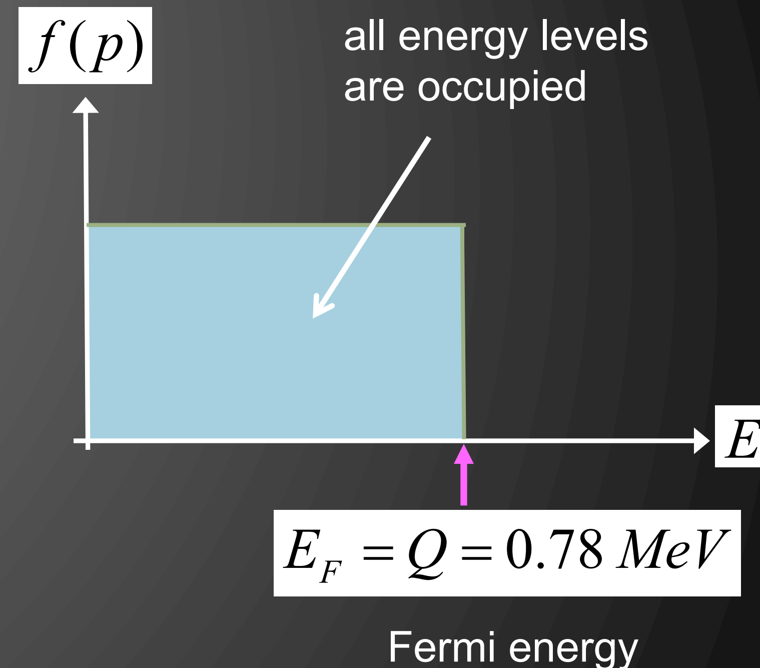
$$E_e^{total} > (m_n - m_p)c^2 = 1.29\ MeV$$

$$(i.e.\ \ E_e^{kin} = E_F > 1.29 - 0.511\ MeV = 0.78\ MeV)$$

electron rest mass

production of neutrons:

$$\Rightarrow \quad inverse\ \beta - decay: \quad p + e^- \rightarrow n + \nu_e$$



The newly formed neutron cannot decay back to a proton b/c all energy levels of the electron with E<Q are all occupied by the degenerate Fermi sea.
This is phenomena is called ***Pauli blocling***

$$n \not\rightarrow p + e^- + \bar{\nu}_e$$

$$(n, p, e^-) - gas: \ \rho_{crit} \approx 10^7\ g\,cm^{-3}$$

Ex.22

Summary

Degenerate Fermi Gases

and applications to simple EoS

- Degenerate matter
- Kinetic theory
  - Distribution function in 6-dim phase-space
  - Fermi-Dirac statistics
  - Number density (n), energy density (u), pressure (P)
- Complete degenerate ideal Fermi gas ($T \rightarrow 0$)
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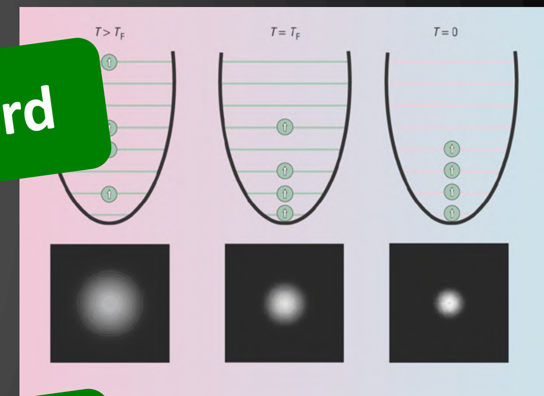
Blackboard



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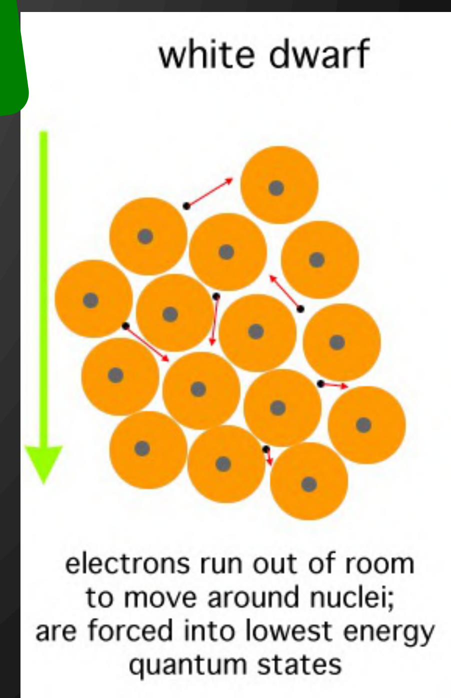
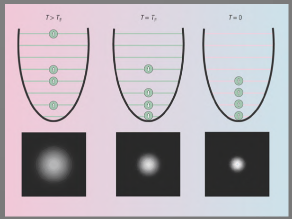
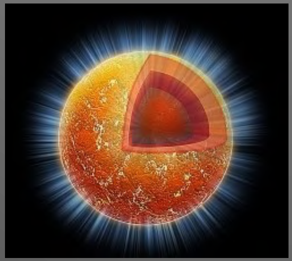
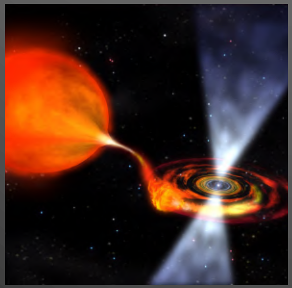


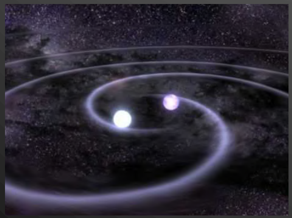
(Chandrasekhar 1931)

Programme









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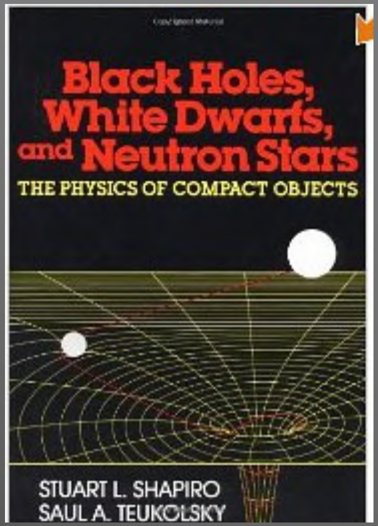
Physics of Compact Objects
week 2



Shapiro & Teukolsky (1983), Wiley-Interscience

Curriculum
- Chapter 2: p.17, 22-29, (29-32), 39-44, Fig.2.2, 2.3, Box 2.1, Table 2.2

Exercises: #20, 22

Next lecture: Structure of White Dwarfs
(Cold Eq. of State Below Neutron Drip). S&T Chapter 2+3.
- Mon. Sept. 18, 08:15-10:00, Aud.2.115